

SET-THEORETIC BLOCKCHAINS

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ABSTRACT. Given a countable model of set theory, we study the structure of its generic multiverse, the collection of its forcing extensions and ground models, ordered by inclusion. Mostowski showed that any finite poset embeds into the generic multiverse while preserving the nonexistence of upper bounds. We obtain several improvements of his result, using what we call the blockchain construction to build generic objects with varying degrees of mutual genericity. The method accommodates certain infinite posets, and we can realize these embeddings via a wide variety of forcing notions, while providing control over lower bounds as well. We also give a generalization to class forcing in the context of second-order set theory, and exhibit some further structure in the generic multiverse, such as the existence of exact pairs.

1. INTRODUCTION

Forcing is the predominant method of building new models of set theory. From a given model we may build up myriad forcing extensions, adding new objects to our models as we proceed; or we may in contrast dig down on a geological tack to various grounds and the mantle, as done recently by Fuchs, the second author, and Reitz [FHR15], stripping away any superfluous forcing that might earlier have been performed. By iterating these two operations in turn, we thereby construct a robust collection of models, all ultimately derived from the initial model and exhibiting a certain family resemblance. Namely, we arrive at the generic multiverse of the original model, first defined by Woodin [Woo11].

1.1. Definition. The *generic multiverse* of a countable transitive¹ model of set theory $M \models \text{ZFC}$ is the smallest collection of models containing M and closed under forcing extensions and grounds. \diamond

We shall simply refer to *the* generic multiverse, although of course the particular multiverse that arises depends on the original model M , or indeed on any of the models in it, for the generic multiverse of a model is an equivalence class in the corresponding partition of the space of all models. But in this article, let it be understood that we start from an arbitrary countable transitive model of set theory M , fixed for the remainder of the paper, and consider the resulting generic multiverse of this model.

Given a definable class Γ of posets, such as CCC or proper posets and so on, one could just as well consider the Γ -generic multiverse, obtained by forcing extensions and ground models using only forcing notions in Γ . The definition of Γ should be reinterpreted in each model anew, with the restriction to CCC posets, say, being read *de dicto* rather than *de re*.

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¹ Transitivity is not actually required for our arguments, but it simplifies the presentation.

Typically, the generic multiverse is of interest because of the models themselves. After all, it arose because of the study of forcing which gives us new and exciting models of set theory to play with. But the generic multiverse is also an interesting object *per se* and as a whole. Just as computability theorists are interested not only in the computational content of particular Turing degrees, but also in the global structure of these degrees, so too can set theorists venture beyond the particular models in the generic multiverse and study the structure as a whole. The aspect of this structure in which we are most interested in this paper is simply the inclusion relation between models.

In work growing out of the modal logic of forcing [HL08, HL13] and set-theoretic geology [FHR15], the second author had inquired whether the inclusion relation in the generic multiverse coincides with the forcing-extension relation (e.g. [Ham16, Question 1]). Since one might arrive from one model to a smaller model by means of a circuitous zig-zag path of successive forcing extensions and grounds, there seemed at first to be no direct reason to expect the smaller model necessarily to be a ground of the original model. Nevertheless, the second author [Ham16, Theorem 3] had proved that if the downward-directed grounds hypothesis DDG was true, then indeed the inclusion relation was the forcing-extension relation in the generic multiverse. In a major result, Usuba [Usu17] subsequently proved that indeed the generic multiverse is downward directed, establishing the following corollary.

1.2. Lemma. *Let N_0 and N_1 be models in the generic multiverse of M . If $N_0 \subseteq N_1$ then N_1 is a forcing extension of N_0 .*

Proof. The proof amounts to [Ham16, Theorem 3] plus Usuba’s result on the DDG [Usu17]. Namely, from the DDG it follows that the collection of forcing extensions of grounds of M is closed under both forcing extensions and grounds, and so this collection of models is precisely the generic multiverse of M . If $N_0 \subseteq N_1$ are among these, then there is some ground N of N_1 such that $N \subseteq N_0 \subseteq N_1$. The model N_0 is therefore intermediate between a ground model and one of its forcing extensions. It follows by the intermediate model theorem (see [Jec03, Lemma 15.43 and the discussion preceding Theorem 16.4]) that N_1 is also a forcing extension of N_0 . \square

The order structure of the generic multiverse has been studied before. For example, Reitz [Rei07] studied the ground axiom, a first-order set-theoretic statement describing the minimal elements in their generic multiverses, and this work subsequently led to the development of set-theoretic geology (see [FHR15]), the study of ground models and “going down” in the generic multiverse. The second author and Löwe had studied the modal logic both of forcing extensions and of grounds [HL08, HL13], and the second author had looked further into issues of upward closure of the generic multiverse in [Ham16].

Turning our glance upward, it is clear that the generic multiverse does not have any maximal elements. However, the upward directedness of the multiverse and the existence of suprema definitely pose interesting questions.

1.3. Definition. Let M be a countable model of ZFC and let \mathcal{E} be a family of forcing extensions of M . We say that the family \mathcal{E} is *amalgamable* (over M) if there is a forcing extension $M[G]$ of M that contains every model in \mathcal{E} . \diamond

We shall similarly say that a family of generic objects is amalgamable if the corresponding family of generic extensions of M is amalgamable.

It follows from Usuba’s results that a collection \mathcal{E} as above is amalgamable precisely when it has an upper bound in the generic multiverse of M . Furthermore, a similar fact holds not just for forcing extensions of M , but throughout the generic multiverse: Any given finite collection of models in the multiverse has an upper bound if and only if the collection is amalgamable over some model in the multiverse.

We should also note that the amalgamability of some family of extensions $\{M[G_i] \mid i \in I\}$, does not require the amalgamating model to have the actual set $\{G_i \mid i \in I\}$ or the associated sequence. Of course, this makes no difference in the case of finite I , but becomes important when discussing the (non)amalgamability of infinite families.

Mostowski [Mos76] essentially showed that any finite poset embeds into the generic multiverse in a way that preserves the nonexistence of upper bounds. For example, this means that, given a countable model M , there are two forcing extensions $M[G]$ and $M[H]$ such that no forcing extension of M contains both $M[G]$ and $M[H]$; in other words, $M[G]$ and $M[H]$ form a nonamalgamable pair. The extensions constructed by Mostowski are all obtained via Cohen forcing, but further work in [Ham16] shows that the nonamalgamability phenomenon is pervasive and is exhibited by a much larger family of forcing notions. In the direction of positive structural results, Fuchs, the second author, and Reitz [FHR15] showed that, while not all chains in the generic multiverse have upper bounds, any countable increasing chain arising from posets of uniformly bounded size does have an upper bound. Taken together, these results suggest that the order structure of the generic multiverse is quite complex.

In this paper we will give a number of improvements to Mostowski's result, strengthening the embedding, generalizing to certain infinite posets, and using forcing notions beyond just Cohen forcing. All these generalizations are obtained using a powerful method we call the *blockchain construction*. The method constructs a family of generic objects by stringing together blocks, each of which conceals a piece of forbidden information. The blocks are furthermore structured in such a way that this secret remains hidden unless we have access to a collection of the generics that was meant to be nonamalgamable. In that case the whole structure of the blocks is revealed, together with the hidden information.

Our main results are the following (see [section 2](#) and [section 3](#) for definitions):

1.4. Theorem. *If a family of sets \mathcal{A} is defined in M by finite obstacles, then (\mathcal{A}, \subseteq) $*$ -embeds into the generic multiverse of M .*

1.5. Theorem. *If a family of sets \mathcal{A} is defined in M by finite obstacles on a set I and $\{\mathbb{P}_i \mid i \in I\}$ is a family of wide posets in M , all the same size and at least as large as I , then (\mathcal{A}, \subseteq) $*$ -embeds into the generic multiverse, with the additional property that each $A \in \mathcal{A}$ maps to a forcing extension of M by the product $\prod_{i \in A} \mathbb{P}_i$.*

We also have versions of this theorem for the class-generic multiverse, generated by class forcings over a model of second-order set theory. (See [section 4](#) for definitions.)

1.6. Theorem. *Let (M, \mathcal{X}) be a countable transitive model of GBC plus the Elementary Class Choice principle. If a family of sets $\mathcal{A} \in M$ is defined in (M, \mathcal{X}) by finite obstacles on a class $I \in \mathcal{X}$ and $\{\mathbb{P}_i \mid i \in I\} \in \mathcal{X}$ is a family of pretame Ord-wide class forcing notions, then (\mathcal{A}, \subseteq) $*$ -embeds into the class-generic multiverse of (M, \mathcal{X}) , with each $A \in \mathcal{A}$ mapping to a forcing extension of (M, \mathcal{X}) by the product $\prod_{i \in A} \mathbb{P}_i$.*

Beyond generalizing Mostowski's result, we also demonstrate some further complexities of the generic multiverse. For example:

1.7. Theorem. *Suppose $M \subseteq M[c_0] \subseteq M[c_1] \subseteq \dots$ is a countable chain where each c_n is a Cohen real over M . Then this chain admits an exact pair and, consequently, does not have a supremum in the generic multiverse.*

2. THE BLOCKCHAIN ARGUMENT—FOLLOWING MOSTOWSKI

In this section we would like to present an adaptation of Mostowski's original argument from [Mos76] to the context of the generic multiverse, as well as an improved version thereof. Our purpose in recounting Mostowski's proof is twofold: first, it will serve as a template for our own proofs going forward, and second, the result seems largely unknown and we would like to highlight it (indeed, we had been unaware of Mostowski's result and found the reference only after having made essentially similar arguments on our own). To be clear, Mostowski was interested in studying what was substantially a multiverse of countable models of second-order set theory with a fixed first-order part, together with the corresponding notion of amalgamability. Nevertheless, the methods apply equally well to the generic multiverse setting (and we will return to the second-order context in [section 4](#)).

2.1. Theorem (Mostowski). *Let $\ell < \omega$ and let \mathcal{A} be a family of subsets of $\{0, 1, \dots, \ell\}$ containing all singletons and closed under subsets. Then there are Cohen reals c_0, c_1, \dots, c_ℓ such that, for any $A \subseteq \{0, 1, \dots, \ell\}$, the family $\{M[c_n] \mid n \in A\}$ is amalgamable precisely when $A \in \mathcal{A}$.*

In the proof we will construct the reals c_n step by step, meeting requirements for amalgamability and nonamalgamability in turn. Since the nonamalgamability steps are the key ones, we first present a simpler construction, where these are the only steps we need to make.

2.2. Proposition. *There are Cohen reals c, d over M such that $M[c]$ and $M[d]$ are nonamalgamable.*

2.3. Definition. Let M be a countable transitive model of ZFC. A real z is *catastrophic* for M if the height of any model of ZFC containing z is greater than that of M . \diamond

For example, any real coding the height of M will definitely be catastrophic for M . The important feature of catastrophic reals is that they cannot appear in any model in the generic multiverse of M . Consequently, our strategy for ensuring nonamalgamability will be to code catastrophic reals between various generics.

Proof of Proposition 2.2. We will build two descending sequences of Cohen conditions p_n and q_n that will each generate their own generic real c and d , and along the way we will also make sure to split a code of a catastrophic real between c and d .

Enumerate the open dense subsets of $\text{Add}(\omega, 1)$ in M as $\langle D_n \mid n < \omega \rangle$ and fix a catastrophic real z for M . Suppose that we have already built conditions p_m, q_m for $m < 2n$ and each pair of conditions p_m, q_m has the same length. At this stage of the construction we are handed the open dense set D_n ; we properly extend the condition on the p -side into D_n and pad the condition on the q -side to the same length with 0s. Finally, extend each of the resulting conditions by appending a 1 and then the bit $z(2n)$. This gives us the conditions p_{2n}, q_{2n} . At the next stage of the construction we do the exact same thing but with the roles of the two conditions switched and append the bit $z(2n + 1)$. We illustrate the construction in Figure 1.

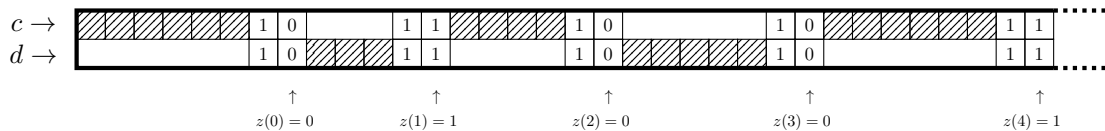


FIGURE 1. Coding z between the two generics. The clear areas are padded with 0s and each point coding a bit of z is preceded by a start marker—a single '1'.

We now let $c := \bigcup_n p_n$ and $d := \bigcup_n q_n$. Clearly both c and d are Cohen reals over M . But note that any model containing both of them would be able to recover z , since the places where we coded z are precisely those places which follow a start marker (i. e. both c and d have a 1 on the previous place—the start marker—and at least one of c and d has a 0 immediately preceding the start marker). This means that $M[c]$ and $M[d]$ cannot be amalgamated, since z is catastrophic for M . \square

The key to this argument was that the two generic reals were built in blocks, and within each block only one of them was permitted to be active, that is, to have nonzero bits. This allowed us to use the places where both were active to mark the endpoints of the blocks and thus the coding points where z was hidden.

The proof of Mostowski's theorem will use the same strategy, building a chain of blocks and only permitting certain columns to be active within each block.

Proof of Theorem 2.1. Fix an enumeration in order type ω of all pairs (A, D) where $A \in \mathcal{A}$ and $D \in M$ is an open dense subset of $\text{Add}(\omega, A)$. Also fix a catastrophic real z for M . We again build descending sequences of conditions, seen as filling in an ω -by- $(\ell + 1)$ matrix with 0s and 1s. The columns of this matrix will grow into the desired generic reals c_k . We will also ensure that at each

step of the construction we have completely filled in the matrix below some row and that there are no entries above this row.

Suppose that we are at some stage in our construction and we are handed an $A \in \mathcal{A}$ and a dense set $D \subseteq \text{Add}(\omega, A)$. We now properly extend the columns with indices in A to collectively meet the dense set D and pad the remaining columns with 0s to make sure that all columns have the same height. Finally, extend each column by appending a 1 and then the next bit of z . We illustrate the first few steps of this construction in Figure 2.

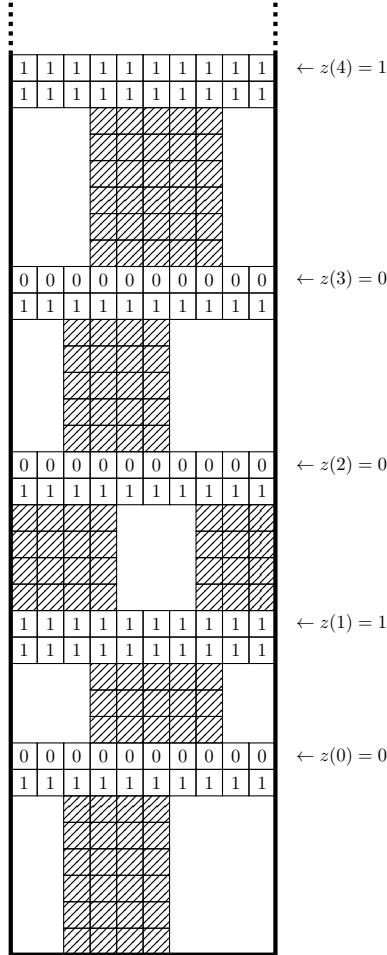


FIGURE 2. The set-theoretic blockchain of Theorem 2.1. Clear areas are again padded with 0s.

This completes the construction and we can let $c_k := \bigcup_n p_k^n$. Given any $A \in \mathcal{A}$, the columns in A were given a chance to meet all of the dense subsets of $\text{Add}(\omega, A)$, and so $\{c_k \mid k \in A\}$ are mutually generic over M , and consequently they are amalgamable.

On the other hand, suppose we are given $\{c_k \mid k \in B\}$ for some $B \notin \mathcal{A}$. Observe that the only places where all of the columns in B are active are exactly the coding points for z , since \mathcal{A} is closed under subsets, so in between the coding points B always has to have a column padded with 0s. It follows that, given the reals c_k for $k \in B$, we can recover the catastrophic real z , which means that the corresponding extensions are not amalgamable. \square

It is not difficult to see that any finite poset embeds into a family of sets of natural numbers closed under subsets in a way that preserves the nonexistence of upper bounds. It therefore follows from Mostowski's theorem that any finite poset embeds into the generic multiverse while preserving nonamalgamability. In the next theorem we shall extend this result to also include certain infinite posets and also to have the embedding respect lower bounds.

2.4. Definition. Let P and Q be posets with least elements 0_P and 0_Q , respectively. An injective map $f: P \rightarrow Q$ is a **-embedding* if:

- $x \leq y \iff f(x) \leq f(y)$ for any $x, y \in P$,
- any finite subset X of P has an upper, or nonzero lower, bound in P if and only if $f[X]$ has an upper, or nonzero lower, bound in Q , respectively. \diamond

It follows from the definition that a *-embedding preserves the least element. Furthermore, the forward direction of the equivalences in the second bullet point easily follows from the fact that f is increasing.

2.5. Definition. A family of sets \mathcal{A} is *defined by a set of obstacles* \mathcal{B} on a set I if \mathcal{A} consists of the subsets of I that do not contain an element of \mathcal{B} . \diamond

Note that any such family is closed under subsets, since if a set avoids all the obstacles, then so does any subset. For a given family \mathcal{A} defined by obstacles, we may assume without loss that $I = \bigcup \mathcal{A}$, simply by replacing I with $\bigcup \mathcal{A}$. This amounts to assuming that \mathcal{A} contains all singletons from I . We may also assume that the obstacle sets are subsets of I and that they all have at least two elements. So we will generally make these additional assumptions without remark. We shall be principally interested in the case where all the obstacles are finite.

2.6. Theorem. *Suppose that a family of sets \mathcal{A} is defined in M by finite obstacles on a set I . Then there are M -generic Cohen reals $\{c_i \mid i \in I\}$, with the following properties:*

- (1) *If $A \in \mathcal{A}$, then $\langle c_i \mid i \in A \rangle$ is generic for $\text{Add}(\omega, A)$ over M .*
- (2) *If $B \in M$, $B \subseteq I$ and $B \notin \mathcal{A}$, then the family $\{M[c_i] \mid i \in B\}$ does not amalgamate in the generic multiverse.*
- (3) *If $A, A' \in \mathcal{A}$ then $M[c_i \mid i \in A] \cap M[c_i \mid i \in A'] = M[c_i \mid i \in A \cap A']$.*

2.7. Definition. Let I be a set and \mathbb{P}_i for $i \in I$ posets. If $p, q \in \prod_{i \in I} \mathbb{P}_i$ and $J \subseteq I$, we say that p is a *J -extension* of q , and write $p \leq_J q$, if $p \leq q$ and $p \upharpoonright_{I \setminus J} = q \upharpoonright_{I \setminus J}$. \diamond

Thus, a J -extension strengthens the condition only on the coordinates in J , leaving the other coordinates untouched. We emphasize that the relation \leq_J is not simply the pullback of the order on $\prod_{i \in J} \mathbb{P}_i$.

2.8. Lemma. *Let $X \subseteq Y$ be sets and let \mathbb{P}_y for $y \in Y$ be posets. Let \mathbb{P}_Y be the product of the \mathbb{P}_y (using whatever support you like) and let \mathbb{P}_X be the subposet corresponding to the coordinates in X . Let χ be a \mathbb{P}_Y -name such that $\mathbb{P}_Y \Vdash \chi \subseteq V[\dot{G} \upharpoonright_X]$ but $\mathbb{P}_Y \nVdash \chi \in V[\dot{G} \upharpoonright_X]$. Fix a condition $q \in \mathbb{P}_Y$ such that $q \Vdash \chi \notin V[\dot{G} \upharpoonright_X]$. Then there is a \mathbb{P}_X -name ρ such that no X -extension of q decides $\rho \in \chi$.*

Proof. First consider the case $X = \emptyset$. So suppose that for every x the condition q decides $\check{x} \in \chi$. Let $S = \{x \mid q \Vdash \check{x} \in \chi\}$. Clearly S is a ground-model set and $q \Vdash \chi = \check{S}$, which contradicts our assumption.

In the general case, observe that

$$q \upharpoonright_X \Vdash (q \upharpoonright_{Y \setminus X} \Vdash \chi \notin V[\dot{G} \upharpoonright_X])$$

Applying the previous case in $V^{\mathbb{P}_X}$, we can find a \mathbb{P}_X -name ρ such that

$$q \upharpoonright_X \Vdash (q \upharpoonright_{Y \setminus X} \text{ does not decide } \rho \in \chi)$$

We claim that this ρ is as desired. So fix some $r \leq_X q$. Then it follows from the above that

$$r \upharpoonright_X \Vdash (q \upharpoonright_{Y \setminus X} \text{ does not decide } \rho \in \chi)$$

but, since $r \upharpoonright_{Y \setminus X} = q \upharpoonright_{Y \setminus X}$, we in fact get

$$r \upharpoonright_X \Vdash (r \upharpoonright_{Y \setminus X} \text{ does not decide } \rho \in \chi)$$

which implies that r does not decide $\rho \in \chi$, as desired. \square

Proof of Theorem 2.6. We proceed with a blockchain argument very much like in the proof of Theorem 2.1. Some extra care is required since we can no longer terminate blocks by putting 1s across an entire row, as this would prevent any infinite collection of the c_i from being mutually generic. Instead we will work with blocks of finite width, and it is precisely the fact that \mathcal{A} is defined by finite obstacles that will allow this approach to succeed.

Let \mathcal{B} be the set of finite obstacles used to define \mathcal{A} . Fix a catastrophic real z for M . We will be filling in an ω -by- I matrix with 0s and 1s, and its columns will grow into the desired Cohen reals c_i . We will ensure at each stage of our construction that what we have constructed so far is a condition in $\text{Add}(\omega, I)$, in effect using finite support in the product. Moreover, we will make sure that this condition is *uniform*, in the sense that all of its nonempty columns will have the same height. Our construction will again be guided by some enumeration of all the relevant requirements in order type ω . We will, moreover, assume that each requirement is listed infinitely often. At each stage the enumeration will give us either

- (1) an $A \in \mathcal{A}$ and an open dense subset of $\text{Add}(\omega, A)$,
- (2) a $B \in \mathcal{B}$, or
- (3) a pair $(A, A') \in \mathcal{A}^2$, together with an $\text{Add}(\omega, A)$ -name $\sigma \in M$ and an $\text{Add}(\omega, A')$ -name $\tau \in M$ for subsets of $M[\dot{G}\upharpoonright_{A \cap A'}]$.

Now suppose that we are at some stage of our construction, p is the entire condition in $\text{Add}(\omega, I)$ we have constructed so far, and we are given an open dense subset $D \subseteq \text{Add}(\omega, A)$ for some $A \in \mathcal{A}$. In this case we simply extend the columns of p in A to meet the dense set D and then pad with 0s, if necessary, to obtain a uniform condition.

If we are given an obstacle $B \in \mathcal{B}$, we extend the columns of p in B by appending a row of 1s and then the bit $z(n)$, where n is the number of times that B has occurred in the enumeration before this step. Note that the result is still a condition in $\text{Add}(\omega, I)$ since B is finite. At the end we pad with 0s to make the condition uniform, if necessary.

Finally, suppose we are given a pair of names σ and τ as above. If $p\upharpoonright_A \Vdash \sigma \in M[\dot{G}\upharpoonright_{A \cap A'}]$ we do nothing in this step and proceed with the rest of the construction. If this is not the case, we will attempt to find an extension q of p that forces $\sigma \neq \tau$. Given our assumption, we can find some $p' \leq_A p$ such that $p'\upharpoonright_A \Vdash \sigma \notin M[\dot{G}\upharpoonright_{A \cap A'}]$. Now apply Lemma 2.8 with $X = A \cap A'$, $Y = A$, $q = p'\upharpoonright_A$, and $\chi = \sigma$ to get a name ρ such that no $(A \cap A')$ -extension of $p'\upharpoonright_A$ decides $\rho \in \sigma$. Pick an A -extension of p' that decides $\rho \in \sigma$ and pad it with 0s to make it uniform; let p'' be the resulting condition. Now consider two cases:

- (1) If $p''\upharpoonright_{A'}$ does not decide $\rho \in \tau$ the same way as $p''\upharpoonright_A$ decides $\rho \in \sigma$, we can find a $q \leq_{A'} p''$ such that $q\upharpoonright_{A'}$ decides it the opposite way. In this case we pad q with 0s to make it uniform, and the resulting condition is the next step in our construction.
- (2) If $p''\upharpoonright_{A'}$ decides $\rho \in \tau$ the same way as $p''\upharpoonright_A$ decides $\rho \in \sigma$, observe that $p'\upharpoonright_A \cup p''\upharpoonright_{A \cap A'}$ is an $(A \cap A')$ -extension of $p'\upharpoonright_A$, so, by our assumption, it does not decide $\rho \in \sigma$. Therefore there must be some $q \leq_A p' \cup (p''\upharpoonright_{A'})$ so that $q\upharpoonright_A$ decides $\rho \in \sigma$ differently than $q\upharpoonright_{A'} \leq p''\upharpoonright_{A'}$ decides $\rho \in \tau$. We take this q to be the next step in our construction, after possibly padding with 0s to make it uniform.

Note that this q is as desired: Since q decides $\rho \in \sigma$ and $\rho \in \tau$ in opposite ways, it definitely forces $\sigma \neq \tau$.

This finishes the construction of the reals c_i . Let us verify that they satisfy all of the properties in the theorem. If $A \in \mathcal{A}$ then, in the course of our construction, we considered every open dense subset of $\text{Add}(\omega, A)$ in M , and we made sure that the columns of our matrix in A met these dense sets. Therefore $\langle c_i \mid i \in A \rangle$ really is generic for $\text{Add}(\omega, A)$.

If $B \notin \mathcal{A}$, then it must contain an obstacle $B' \in \mathcal{B}$, and we handled this B' infinitely often during our construction, each time adding a row of 1s across the columns in B' and the next bit of z . Furthermore, we made sure in our construction to only add a row of 1s across a set of columns if the set was indeed an obstacle set (in which case we also coded a bit of z) or if it was covered by an element of \mathcal{A} . Since the latter is not the case for B' , we can recover z from the reals $\{c_i \mid i \in B'\}$, and so the family $\{M[c_i] \mid i \in B'\}$ is not amalgamable and therefore neither is $\{M[c_i] \mid i \in B\}$.

Finally, suppose that $A, A' \in \mathcal{A}$ and we want to see that

$$M[c_i \mid i \in A] \cap M[c_i \mid i \in A'] \subseteq M[c_i \mid i \in A \cap A']$$

Let x be an element of this intersection; we may assume by \in -induction that $x \subseteq M[c_i \mid i \in A \cap A']$. Pick an $\text{Add}(\omega, A)$ -name σ and an $\text{Add}(\omega, A')$ -name τ for x . But we were handed this pair of names at some stage of our construction. At that point it must have been the case that the condition we had forced $\sigma \in M[\dot{G} \upharpoonright_{A \cap A'}]$, since otherwise we would have made sure that σ and τ would be interpreted as different sets. But this means that $x \in M[c_i \mid i \in A \cap A']$, as required. \square

2.9. Theorem. *Let I and \mathcal{A} be as in [Theorem 2.6](#). Then (\mathcal{A}, \subseteq) $*$ -embeds into the generic multiverse given by posets of the form $\text{Add}(\omega, X)$ for a set X (and mapping \emptyset to M).*

Proof. Let $\{c_i \mid i \in I\}$ be the Cohen reals given by [Theorem 2.6](#). We define a map on \mathcal{A} by sending $A \in \mathcal{A}$ to $M[c_i \mid i \in A]$. This map is clearly increasing. Let us check that it is also a $*$ -embedding.

Suppose that $M[c_i \mid i \in A] \subseteq M[c_i \mid i \in A']$ for some $A, A' \in \mathcal{A}$. It follows that the family $\{M[c_i \mid i \in A \cup A']\}$ is amalgamable, so $A \cup A' \in \mathcal{A}$. In particular, $\langle c_i \mid i \in A \cup A' \rangle$ are mutually generic over M . Now suppose that there is some $\ell \in A \setminus A'$. Then c_ℓ would have to be generic over $M[c_i \mid i \in A'] \supseteq M[c_i \mid i \in A]$, which is impossible. Therefore $A \subseteq A'$.

Now suppose that X is a finite subset of \mathcal{A} and that $\{M[c_i \mid i \in A] \mid A \in X\}$ is amalgamable. It follows as above that $\bigcup X \in \mathcal{A}$, so X does have an upper bound in \mathcal{A} .

Finally, with X as above, suppose that $\{M[c_i \mid i \in A] \mid A \in X\}$ has a lower bound in the generic multiverse strictly above M . But recall that one of the properties of the Cohen reals constructed in [Theorem 2.6](#) is that $\bigcap_{A \in X} M[c_i \mid i \in A] = M[c_i \mid i \in \bigcap X]$. It follows that $\bigcap X$ is nonempty, and so it is a nonzero lower bound for X in \mathcal{A} . \square

The theorem is actually slightly stronger than stated, since we obtain nonamalgamability in the entire generic multiverse and not just in the restricted multiverse given by adding Cohen reals. On a related note, we can tweak the real z that we coded between the generics to achieve different kinds of nonamalgamability. For example, we could take z to be a sufficiently generic random real over M , in which case the models given will be nonamalgamable in the Cohen multiverse, but amalgamable in, say, the CCC-multiverse.

2.10. Corollary. *Any finite poset with a least element $*$ -embeds into the generic multiverse. Moreover, any finite meet-semilattice $*$ -embeds as such into the generic multiverse.*

Proof. It is easy to check that the composition of $*$ -embeddings is itself a $*$ -embedding, so, in view of [Theorem 2.9](#), we only need to check that any finite poset P $*$ -embeds into a family of sets \mathcal{A} as in [Theorem 2.6](#). We can take \mathcal{A} to consist of all sets of the form $p \downarrow := \{q \in P \mid 0_P < q \leq p\}$ and their subsets, where $p \in P$ and 0_P is the least element of P , and it is straightforward to see that the map $p \mapsto p \downarrow$ is a $*$ -embedding. If P is in addition a meet-semilattice, we can furthermore observe that both the map $p \mapsto p \downarrow$ and the $*$ -embedding from [Theorem 2.9](#) preserve meets. \square

3. GENERALIZING TO WIDE FORCING

The blockchain arguments given in the previous section clearly generalize to other forcing notions where conditions are some kind of bounded sequence, such as $\text{Add}(\kappa, 1)$ or $\text{Coll}(\kappa, \lambda)$. The next question we are interested in is whether the nonamalgamability phenomenon (and various $*$ -embeddings) can be realized using an even larger variety of forcing notions, and even possibly while using several nonisomorphic posets at once.

The obvious obstacle to our grand ambition is that there are always pairs of posets such that any two generic extensions of M by these posets amalgamate: just take $\text{Add}(\omega, 1)$ and $\text{Add}(\omega_1, 1)$, for example. Retreating a bit to just consider one poset at a time, the second author showed in [\[Ham16\]](#) that, under \diamond , there is a Suslin tree T exhibiting *automatic mutual genericity*, meaning that any two generic filters for T are mutually generic. Any two extensions by such forcing are therefore amalgamable. The question whether posets exhibiting automatic mutual genericity always exist

is open, although Ben-Neria and Chen managed to show that the Suslin tree used by the second author is essentially the only CCC example (see [Ham15]).

Skirting these counterexamples, the second author introduced the notion of a wide poset and proved the nonamalgamation theorem for wide forcing.

3.1. Definition ([Ham16]). A poset \mathbb{P} is *wide* if there is, for any $p \in \mathbb{P}$, a maximal antichain in \mathbb{P} below p of size $|\mathbb{P}|$. \diamond

For example, the Cohen poset $\text{Add}(\omega, 1)$ is wide, as are collapse posets $\text{Coll}(\kappa, \lambda)$, Sacks forcing, Mathias forcing, etc. It is readily seen that the class of wide posets is closed under finite products, finite lottery sums, provided that the factors are all of the same size, and finite iterations (although some care is needed in the specific presentation of the iteration).

The second author went on to show that, given posets \mathbb{P}_0 and \mathbb{P}_1 that are wide and of equal size in M , there are generic filters G_0 and G_1 such that the extensions $M[G_0]$ and $M[G_1]$ do not amalgamate. Let us now generalize this argument to prove a version of [Theorem 2.6](#) in the setting of wide posets.

3.2. Theorem. *Suppose that a family of sets \mathcal{A} is defined in M by finite obstacles on a set I and $\{\mathbb{P}_i \mid i \in I\} \in M$ is a family of wide posets in M , all the same size $\kappa \geq |I|$. Then there are generic filters $G_i \subseteq \mathbb{P}_i$ over M with the following properties:*

- (1) *If $A \in \mathcal{A}$ then $\prod_{i \in A} G_i$ is generic for $\prod_{i \in A} \mathbb{P}_i$ over M .*
- (2) *If $B \in M$, $B \subseteq I$ and $B \notin \mathcal{A}$, then the family $\{M[G_i] \mid i \in B\}$ does not amalgamate in the generic multiverse.*
- (3) *If $A^0, A^1 \in \mathcal{A}$ then $M[\prod_{i \in A^0} G_i] \cap M[\prod_{i \in A^1} G_i] = M[\prod_{i \in A^0 \cap A^1} G_i]$.*

The products above are to be seen as subposets of the product $\prod_{i \in I} \mathbb{P}_i$ which may use supports from an arbitrary ideal in M extending the finite ideal on I .

Proof. For X a subset of I we write $\mathbb{P}_X = \prod_{i \in X} \mathbb{P}_i$ and similarly for G_X . For each $i \in I$ we fix an enumeration of \mathbb{P}_i in order type κ in M . Since the \mathbb{P}_i are wide in M , we can also find for each $p \in \mathbb{P}_i$ a maximal antichain $Z(p) \subseteq \mathbb{P}_i$ below p of size κ , together with an enumeration in order type κ . Finally, fix a catastrophic real z for M .

As in the blockchain construction of [section 2](#) we will undertake a construction of a descending sequence of conditions $p_n = \langle p_n^i \mid i \in I \rangle \in \prod_{i \in I} \mathbb{P}_i$ whose restriction to each coordinate i will generate the generic filter G_i . We will be guided by some enumeration of all the relevant parameters in order type ω . We arrange matters such that at each stage of the construction we are given the following: two sets $A^0, A^1 \in \mathcal{A}$, a dense subset $D \subseteq \mathbb{P}_{A^0}$ in M , and a \mathbb{P}_{A^0} -name $\sigma \in M$ and a \mathbb{P}_{A^1} -name $\tau \in M$, both for subsets of $M[\dot{G}_{A^0 \cap A^1}]$ (of course, we want to run through all the possible combinations of these parameters).

Before we start the construction, let us briefly describe the idea. The key property of the wide posets \mathbb{P}_i will be that, given a condition $p \in \mathbb{P}_i$, a generic G_i containing p picks out a unique element of the antichain $Z(p)$, which amounts to, in view of our fixed enumeration, a particular ordinal below κ . We could, for example, use this mechanism to encode the bits of the catastrophic real z as we were building our sequence of conditions p_n . This approach would be similar to what we did in the proof of [Theorem 2.6](#). However, since the structure of the posets \mathbb{P}_i may be more complicated than that of Cohen forcing, we might, along the way, lose track of the *coding points*—those conditions whose corresponding antichains we should consult to decode z . To remedy this, we will keep track of the coding points used at each stage and code not only the catastrophic real z but also the next set of coding points. In addition, at various points in the construction we will want to record one of three additional special symbols (say \sharp , \flat , and \natural) which will serve to mark which case of the construction we are in. These special symbols will be used in the decoding process to signal which generic to look at to recover further information.

We start the construction by letting p_0^i be the top condition of \mathbb{P}_i for each $i \in I$. Now suppose that we have constructed p_n and are given $A_n^0, A_n^1 \in \mathcal{A}$, a dense $D \subseteq \mathbb{P}_{A_n^0}$, and names σ and τ . Let B_n^0 and B_n^1 be the sets of the first n many elements of $I \setminus A_n^0$ and $I \setminus A_n^1$, respectively, in some

fixed enumeration of I in V in order type ω . We also assume that we have, along the way, built a sequence of conditions $c_m \in \mathbb{P}_I$ which are the coding points used at stage m . To be precise, for each $m < n$ and each $i \in B_m^0$ we assume that $p_{m+1}^i \leq c_m^i$ is the condition in the antichain below c_m^i whose index codes: the sets B_{m+1}^0 and B_{m+1}^1 , the bit $z(n)$, and the sequence $\langle c_m^j \mid j \in B_{m+1}^0 \rangle$ (note that all of these are essentially finite subsets of κ , so it makes sense to code them together by a single ordinal below κ). If we additionally assume that $p_{m+1}^i = c_m^i$ for $i \notin B_m^0$, then by knowing $c_m \upharpoonright_{B_m^0}$ and a generic G_i for some $i \in B_m^0$, we can recover $p_{m+1} \upharpoonright_{B_{m+1}^0}$.

Returning to our construction at stage n , it remains for us to build $c_n \leq p_n$ in such a way that $c_n \upharpoonright_{B_n^0}$ will be recoverable from $p_n \upharpoonright_{B_n^0}$ and a well-chosen generic. First of all, we can find $q_n \leq_{A_n^0} p_n$ such that $q_n \upharpoonright_{A_n^0} \in D$ and q_n decides whether $\sigma \in M[\dot{G}_{A_n^0 \cap A_n^1}]$. We now have a small tree of cases to consider.

Case 1: Suppose that $q_n \Vdash \sigma \in M[\dot{G}_{A_n^0 \cap A_n^1}]$. In this case we just let $c_n \leq q_n$ be the extension obtained by coding the symbol \sharp below the conditions on the coordinates $i \in B_n^0$ and keeping the other coordinates. In particular, since B_n^0 is disjoint from A_n^0 , once we know $p_n \upharpoonright_{B_n^0}$ and a generic G_i for some $i \in B_n^0$, we can recover $c_n \upharpoonright_{B_n^0}$.

Case 2: Suppose that $q_n \Vdash \sigma \notin M[\dot{G}_{A_n^0 \cap A_n^1}]$. In this case we will attempt to find a condition c_n which forces $\sigma \neq \tau$. We can apply [Lemma 2.8](#) and get a name ρ for an element of $M[\dot{G}_{A_n^0 \cap A_n^1}]$ such that no $(A_n^0 \cap A_n^1)$ -extension of q_n decides whether $\rho \in \sigma$. It will be this ρ that we will try to use to distinguish σ and τ in the extension. We find ourselves in one of two subcases.

Case 2a: Suppose that q_n decides whether $\rho \in \tau$. We then let $q'_n \leq_{A_n^0} q_n$ be any extension which decides $\rho \in \sigma$ in the opposite way. After this we again let $c_n \leq q'_n$ be the extension obtained by coding the symbol \sharp below the conditions on the coordinates $i \in B_n^0$ and keeping the other coordinates. Just as before, knowing $p_n \upharpoonright_{B_n^0}$ and some G_i for $i \in B_n^0$ allows us to recover $c_n \upharpoonright_{B_n^0}$.

Case 2b: Suppose that q_n does not decide whether $\rho \in \tau$. We first find $p'_n \leq_{A_n^0} q_n$ which forces $\rho \in \sigma$ and then define $q'_n \leq p'_n$ by coding the sequence $p'_n \upharpoonright_{B_n^1}$, together with the symbol \flat , below each condition p_n^i for $i \in B_n^0$ and fixing the other coordinates. Observe that, at this point, we can recover $q'_n \upharpoonright_{B_n^0 \cup B_n^1}$ from $p_n \upharpoonright_{B_n^0}$ and some generic G_i for $i \in B_n^0$. But we are not yet finished, and we must ask whether we can commit to forcing $\rho \notin \tau$ or whether we have to roll back some mistakes.

Case 2b (i): Suppose that $q'_n \Vdash \rho \in \tau$. In this case we find $p''_n \leq_{A_n^1} q'_n$ which forces that $\rho \notin \tau$ and then let c_n be the condition obtained from p''_n by coding the sequence $p''_n \upharpoonright_{B_n^0}$ below the conditions on the coordinates $i \in B_n^1$ and fixing the other coordinates. Note that $c_n \upharpoonright_{B_n^0}$ can be recovered from $q'_n \upharpoonright_{B_n^1}$ and some generic G_i for $i \in B_n^1$.

Case 2b (ii): Suppose that $q'_n \Vdash \rho \in \tau$. Recall that we also have $q'_n \Vdash \rho \in \sigma$, so this does not seem to be a good way to go if we want to end up forcing $\sigma \neq \tau$. So we are going to try to undo past mistakes and work instead with the condition $q''_n = q_n \upharpoonright_{A_n^0 \setminus A_n^1} \cup q'_n \upharpoonright_{I \setminus (A_n^0 \setminus A_n^1)}$. The key property of this condition is that it still forces $\rho \in \tau$ but it does not decide $\rho \in \sigma$ since it is essentially an $(A_n^0 \cap A_n^1)$ -extension of q_n . Therefore we can find $p'''_n \leq_{A_n^0} q''_n$ which forces $\rho \notin \sigma$ and then let $c_n \leq p'''_n$ be the condition obtained by coding the sequence $p'''_n \upharpoonright_{B_n^0}$, together with the symbol \flat (cancelling out the \flat from before), below the conditions on the coordinates $i \in B_n^0$ and keeping the other coordinates. Then we can recover $c_n \upharpoonright_{B_n^0}$ from $q'_n \upharpoonright_{B_n^0}$ and some generic G_i for $i \in B_n^0$.

This finishes the construction of c_n and p_{n+1} . It still remains for us to show that the generic filters G_i obtained this way have the desired properties. Of these, property (1) is easiest, since it is clear that we have ensured genericity by running through all the dense sets corresponding to any particular $A \in \mathcal{A}$. Property (3) is also not difficult to see. Suppose that we are given an element $x \in M[G_{A^0}] \cap M[G_{A^1}]$ for some $A^0, A^1 \in \mathcal{A}$ and we wish to show that $x \in M[G_{A^0 \cap A^1}]$. Let σ and τ be the \mathbb{P}_{A^0} - and \mathbb{P}_{A^1} -names for x , respectively. We may assume, by \in -induction, that x is in fact a subset of $M[G_{A^0 \cap A^1}]$. These two names were considered at some stage of the construction, at which time one of two things occurred: Either it was forced at that stage that $\sigma \in M[G_{A^0 \cap A^1}]$, or we arranged matters so that it was forced that $\sigma \neq \tau$. But the second option could not have happened, since the two names both evaluate to x . It therefore follows that $x \in M[G_{A^0 \cap A^1}]$.

Finally we turn to property (2). Suppose that we have a set $B \in M$ which does not belong to \mathcal{A} and we wish to show that the family $\{M[G_i] \mid i \in B\}$ does not amalgamate. Since \mathcal{A} is defined by finite obstacles, it suffices to show this for the finite obstacle sets B . We will argue that, given the filter G_B and some parameters in M , we can recover the catastrophic real z , which prevents amalgamability. Since B is finite, there is some n so that for all larger m we have $B \subseteq A_m^0 \cup B_m^0$ and $B \subseteq A_m^1 \cup B_m^1$. Note that also, since \mathcal{A} is closed under subsets, the intersections $B \cap B_m^0$ and $B \cap B_m^1$ are nonempty. We may assume that we already know the part of z up to n , as well as B_n^0, B_n^1 , and $p_n \upharpoonright_{B_n^0}$, and we are looking to recover $z(n), B_{n+1}^0, B_{n+1}^1$, and $p_{n+1} \upharpoonright_{B_{n+1}^0}$.

Fix indices $i^0 \in B \cap B_n^0$ and $i^1 \in B \cap B_n^1$. First we check to see what symbol is coded at the place where the filter G_{i^0} meets the antichain $Z(p_n^{i^0})$. If we see a \sharp , we know that we were either in Case 1 or 2a at this stage of the construction and we can immediately recover $c_n \upharpoonright_{B_n^0}$, and this condition, together with G_{i^0} again, codes all the information we required. Similarly, if we see a \flat , we know that the construction went through Case 2b. We should compute $q'_n \upharpoonright_{B_n^0 \cup B_n^1}$, as described above, and see whether the \flat is followed by a \natural : if so, we know that we proceeded through Case 2b (ii) and we use G_{i^0} to recover $c_n \upharpoonright_{B_n^0}$, and if not, we proceeded through Case 2b (i) and we use G_{i^1} in the same way. \square

In the exact same way as we proved [Theorem 2.9](#) using [Theorem 2.6](#), we can use [Theorem 3.2](#) to prove the following.

3.3. Theorem. *Let I, \mathcal{A} , and the posets \mathbb{P}_i be as in [Theorem 3.2](#). Then (\mathcal{A}, \subseteq) $*$ -embeds into the generic multiverse given by products of the posets \mathbb{P}_i (and mapping \emptyset to M).*

4. (NON-)AMALGAMABILITY IN THE CLASS-GENERIC MULTIVERSE

In this section we turn our attention to the context of different possible collections of classes for a fixed countable model of ZFC, the original context in which Mostowski [[Mos76](#)] was interested. Specifically, he was interested in the structure of models of Gödel–Bernays set theory whose first-order parts are all the same countable transitive model of ZFC. He investigated the possible patterns of (non)amalgamability of these models. Just as we generalized his argument in the set forcing case, we can do the same for class forcing. The results of this section are an adaptation of the results about wide forcings in [section 3](#).

We will treat models of second-order set theory as two-sorted structures $(M, \mathcal{X}, \in^{(M, \mathcal{X})})$ with sets or first-order part M and classes or second-order part \mathcal{X} . We will suppress writing the membership relation, referring simply to (M, \mathcal{X}) . We may always assume that \mathcal{X} is a collection of subsets of M . If M is transitive with the true \in as its membership relation, then so is \mathcal{X} and the set–set and set–class membership relations of (M, \mathcal{X}) are the true \in .

4.1. Definition. Gödel–Bernays set theory GBC with the axiom of Global Choice is the second-order set theory axiomatized by ZFC for sets, Class Extensionality, Class Replacement, Global Choice (in the form there is a class bijection $\text{Ord} \rightarrow V$), and Elementary Comprehension (i. e. Comprehension for formulae with only set quantifiers (but allowing class parameters)). Dropping Powerset from the axioms for sets gives the theory GBC^- .² \diamond

To make the argument work we will need a slight strengthening of GBC in the ground model.

4.2. Definition. Elementary Class Choice ECC is the schema asserting that if for every set there is a class witnessing some first-order property, then there is a single class coding witnesses for each set. Formally, let $\phi(x, Y, A)$ be a first-order formula, possibly with a set or class parameter A . The instance of ECC for $\phi(x, Y, A)$ asserts that if for every set x there is a class Y such that $\phi(x, Y, A)$ then there is a class C such that for every set x we have that $\phi(x, (C)_x, A)$ holds, where $(C)_x = \{y : (x, y) \in C\}$ is the x -th slice of C . \diamond

² In the absence of Powerset we have that Collection is stronger than Replacement [[Zar96](#)]. We want GBC^- to be axiomatized with Collection, as the version with only Replacement is badly behaved [[GHJ16](#)]. Also, the various forms of Global Choice require Powerset to prove their equivalence. Without Powerset, the existence of a class bijection $\text{Ord} \rightarrow V$ is the strongest. We will need this strongest form of Global Choice in [Theorem 4.7](#).

4.3. **Theorem** (Gitman–Hamkins [GH]). *Kelley–Morse set theory³ does not prove ECC.*

4.4. **Theorem** (Williams [Wil18, Corollary 2.48]). *If (M, \mathcal{X}) is a model of $\text{GBC}^- + \text{ETR}$ then there is $\mathcal{Y} \subseteq \mathcal{X}$ second-order definable such that (M, \mathcal{Y}) is a model of $\text{GBC}^- + \text{ETR} + \text{ECC}$.⁴ As a consequence, the following pairs of theories are equiconsistent:*

- $\text{GBC} + \text{ETR}$ and $\text{GBC} + \text{ETR} + \text{ECC}$.
- $\text{GBC}^- + \text{ETR}$ and $\text{GBC}^- + \text{ETR} + \text{ECC}$.

Next we need a definition of wide forcing specialized to the context of class forcing.

4.5. **Definition.** A class forcing notion \mathbb{P} is *Ord-cc* (synonymously, *has the Ord-chain condition*) if every antichain of \mathbb{P} is set-sized. \diamond

4.6. **Definition.** A class forcing notion \mathbb{P} is *Ord-wide* if for every condition $p \in \mathbb{P}$ we have that \mathbb{P} restricted below p is not Ord-cc. That is, below any condition in \mathbb{P} there is a proper-class-sized antichain. \diamond

For the generic multiverse of a model of second-order set theory we want to confine the collection of forcing notions we allow to only the pretame forcings.⁵ The reason is two-fold. First, pretameness is equivalent to the preservation of GBC^- [Sta84, Fri00]. Indeed, if a class forcing is not pretame then any forcing extension by it will fail to satisfy Replacement [HKS18]. This is problematic for nonamalgamability arguments, as it is Replacement which allows us to argue that a forcing extension of M cannot contain a catastrophic real for M . Second, GBC is not strong enough to prove that every nonpretame forcing notion admits a forcing relation [HKL⁺16]. With this in mind, say that the *class-generic multiverse* of a countable transitive model of second-order set theory (M, \mathcal{X}) is the smallest collection of models containing (M, \mathcal{X}) and closed under grounds and extensions by pretame class forcings. Similar to the set-forcing case, a family of universes in the generic multiverse amalgamates if there is some universe in the generic multiverse which contains all of them.

Given a class forcing notion $\mathbb{P} \in \mathcal{X}$ and a filter $G \subseteq \mathbb{P}$ generic over (M, \mathcal{X}) , meaning that G meets every dense subclass of \mathbb{P} in \mathcal{X} , we will write $(M, \mathcal{X})[G]$ for the generic extension of (M, \mathcal{X}) by G . This is defined in the usual way, with the sets of the extension being the interpretations of set \mathbb{P} -names in M and the classes of the extension being the interpretations of class \mathbb{P} -names in \mathcal{X} . If \mathbb{P} is pretame then it follows from the preservation of GBC^- that the sets in the extension are precisely the classes which are subclasses of sets.

We wish to emphasize that the geology for the class-generic multiverse is not as nice as that for the set-generic multiverse. For example, to argue in Lemma 1.2 that inclusion in the (set) generic multiverse is the same as being a forcing extension, we appealed to the intermediate model theorem. But it is a folklore result that the intermediate model theorem can fail for (tame) class forcings. Nevertheless, these difficulties will not concern us here. We are looking only at forcing extensions of a fixed model, not forcing extensions of grounds of forcing extensions of ... And our method of ensuring nonamalgamability—coding a catastrophic real—rules out amalgamability in the broader generic multiverse, not just among forcing extensions of a fixed model.

The primary concern is with models of GBC , but our argument nowhere uses the axiom of powerset, so we state the theorem in a more general setting.

4.7. **Theorem.** *Assume (M, \mathcal{X}) is a countable transitive model of $\text{GBC}^- + \text{ECC}$, meaning that both M and \mathcal{X} are countable, and suppose a family of sets \mathcal{A} is defined in (M, \mathcal{X}) by finite obstacles on a class $I \in \mathcal{X}$. Let $\{\mathbb{P}_i \mid i \in I\} \in \mathcal{X}$ be a family of Ord-wide forcing notions in (M, \mathcal{X}) .⁶ Then there are $G_i \subseteq \mathbb{P}_i$ generic over (M, \mathcal{X}) satisfying the following three properties.*

³ Recall that Kelley–Morse set theory is axiomatized by the axioms of GBC plus the full Comprehension schema for all formulae, even those with class quantifiers.

⁴ See [GH17] for a definition of Elementary Transfinite Recursion ETR , originally due to Fujimoto [Fuj12]. The theory $\text{GBC} + \text{ETR}$ is stronger than GBC in consistency strength, but weaker than Kelley–Morse set theory and indeed weaker than $\text{GBC} + \Pi_1^1\text{-Comprehension}$.

⁵ See [Fri00] for a definition of pretameness.

⁶ To clarify: the \mathbb{P}_i are proper classes so the literal collection of all of them is too high in rank to be a class. But we can code this collection as a single class via pairs, say using the class $C = \{(i, p) : i \in I \text{ and } p \in \mathbb{P}_i\}$ so that

- (1) If $A \in \mathcal{A}$, then $\prod_{i \in A} G_i$ is generic for $\prod_{i \in A} \mathbb{P}_A$ over (M, \mathcal{X}) .
- (2) If $B \subseteq I$ is in \mathcal{X} with $B \notin \mathcal{A}$, then the family $\{(M, \mathcal{X})[G_i] \mid i \in B\}$ does not amalgamate in the generic multiverse.
- (3) If $A^0, A^1 \in \mathcal{A}$ then $M[\prod_{i \in A^0} G_i] \cap M[\prod_{i \in A^1} G_i] = M[\prod_{i \in A^0 \cap A^1} G_i]$.

The products above are to be seen as subposets of the product $\prod_{i \in I} \mathbb{P}_i$, which may use supports from any ideal in (M, \mathcal{X}) which extends the finite ideal.

Proof sketch. This is proved much the same as [Theorem 3.2](#). We give a sketch of the proof here, making clear where ECC and the strong form of Global Choice are used.

Fix the following notation: For classes $A \subseteq I$ let $\mathbb{P}_A = \prod_{i \in A} \mathbb{P}_i$ and similarly for G_A . By Global Choice fix an enumeration of M in order type Ord^M . Using ECC, pick for each $p \in \mathbb{P}$ a proper-class-sized ($= \text{Ord}^M$ -sized) maximal antichain $Z(p) \subseteq \mathbb{P}_i$ below p . Externally to (M, \mathcal{X}) , fix z a catastrophic real for M , an enumeration of I in order type ω , and an enumeration of all the relevant parameters from M in order type ω . We will build a descending ω -sequence of conditions $p_n = \langle p_n^i \mid i \in I \rangle \in \mathbb{P}_I$ whose restrictions to the i -th coordinate will generate G_i . At each stage in the construction we deal with $A^0, A^1 \in \mathcal{A}$, a dense subclass D of \mathbb{P}_{A^0} in \mathcal{X} , a \mathbb{P}_{A^0} -name $\sigma \in \mathcal{X}$, and a \mathbb{P}_{A^1} -name $\tau \in \mathcal{X}$, both for subsets or subclasses of $(M, \mathcal{X})[G_{A^0 \cap A^1}]$.

As before, each G_i is determined by a descending sequence of conditions $\langle p_i \mid i \in \omega \rangle$. We ensure that G_A is generic for $A \in \mathcal{A}$ by meeting every dense subclass. And the key point of our construction will be to find *coding points* which can then be used to guide the decoding process if we are given generics G_i for all i in some bad subset of I which is not in \mathcal{A} . In light of the global well-order, any element of M can be coded by the index of a condition in the proper-class-sized antichain $Z(p)$ below p . (This uses that the global well-order has ordertype Ord .) As before, we start with p_0^i being the top condition in \mathbb{P}_i . And the construction proceeds from stage n to stage $n+1$ the same as before, breaking down into the same tree of cases.

Having carried out the construction, we can verify (1), (2), and (3) the same as before. \square

For specific class forcings, we may get the same result assuming neither Global Choice nor Elementary Class Choice. Suppose, as an example, that each \mathbb{P}_i is $\text{Add}(\text{Ord}, 1)$, the class of functions from an ordinal to $\{0, 1\}$, ordered by reverse inclusion. Then we can code the catastrophic real directly, as in the argument for [theorem 2.1](#). Indeed, this was the forcing notion Mostowski used in his original investigations.

And once again, the exact same way as we proved [Theorem 2.9](#) using [Theorem 2.6](#), and [Theorem 3.3](#) using [Theorem 3.2](#), we can derive the following as a corollary.

4.8. Corollary. *Let (M, \mathcal{X}) , I , \mathcal{A} , and the posets \mathbb{P}_i be as in [Theorem 4.7](#). Then (\mathcal{A}, \subseteq) $*$ -embeds into the class-generic multiverse of (M, \mathcal{X}) given by products of the posets \mathbb{P}_i (and mapping \emptyset to (M, \mathcal{X})).*

5. SURGERY AND THE MUTABLE BLOCKCHAIN

In this section we wish to revisit the results from [section 2](#) and present an alternative proof of Mostowski's [Theorem 2.1](#). While the obtained result is substantially the same, we believe that the method we employ, *surgery*, is quite powerful and can be used to answer other questions in the area. The main difference between Mostowski's argument (and the derived blockchain construction) and the surgery method we are about to give is that the blockchain argument creates the required generic objects (and thus the $*$ -embeddings) completely anew, whereas with the surgery approach we will be able to start with generic objects given in advance and modify them in such a way that they realize the desired $*$ -embedding.

$\mathbb{P}_i = \{p : (i, p) \in C\}$ is the i -th slice of C . And we must do something similar for sequences of classes, as we will use below.

We remind the reader that, in this section, we work exclusively with the version of Cohen forcing $\text{Add}(\omega, 1)$ whose conditions are finite binary strings. We will comment later why other presentations of this forcing notion are not suitable.

The following definition introduces a useful way of surgically modifying one function to match another one on a subset of its domain.

5.1. Definition. Let f and g be (partial) functions defined on ω and let $A \subseteq \text{dom}(f)$ have the same cardinality as $\text{dom}(g)$. Let $e: A \rightarrow \text{dom}(g)$ be the unique order-preserving bijection. The *graft* of g onto f on A is the function $f \lambda_A g$, defined on $\text{dom}(f)$ as:

$$(f \lambda_A g)(x) := \begin{cases} f(x) & x \notin A \\ g(e(x)) & x \in A \end{cases}$$

In other words, the graft $f \lambda_A g$ is the result of replacing the values of f on A with the corresponding values of g . If $A = \text{dom}(g)$, we omit it and just write $f \lambda g$. \diamond

5.2. Definition. Let p be a condition in $\text{Add}(\omega, 1)$, let D be a dense open subset of this poset, and let n be a natural number. We say that p is (n, D) -*immune* if any condition q satisfying $q(i) = p(i)$ for all $i \geq n$ is in D . In other words, a condition is (n, D) -immune if any modification of it on the coordinates below n results in a condition in D .

Given conditions p and q , we say that q is a D -*immunization* of p if $q \leq p$ and q is $(|p|, D)$ -immune. \diamond

We should note that any (n, D) -immune condition is itself already in D .

5.3. Proposition. *Let D be a dense open subset of $\text{Add}(\omega, 1)$. Any condition $p \in \text{Add}(\omega, 1)$ has a D -immunization.*

Proof. Fix a condition p . Since we are working with conditions that are binary sequences, there are only finitely many conditions of length $|p|$, and we enumerate them as $\langle q_i \mid i < N \rangle$. We will build a descending sequence of increasingly immune conditions p_i for $i \leq N$. Start by letting $p_0 := p$. In general, given p_i , we first let \bar{p}_{i+1} be an extension of $p_i \lambda q_i$ inside D and then define $p_{i+1} := \bar{p}_{i+1} \lambda p$. It is then clear that p_N is a D -immunization of p . \square

The existence of immunizations is the main reason why we need to work with the binary sequence version of Cohen forcing in this section. Working instead with conditions as finite sequences of natural numbers, for example, it is not hard to come up with a dense open set D such that no condition at all is even $(1, D)$ -immune.

The utility of immune conditions is that they admit limited amounts of surgery, while remaining in a given dense set. The plan is then to build a generic filter from a sequence of increasingly immune conditions with the hope that the resulting Cohen real will be able to withstand surgery on unboundedly many coordinates while remaining generic. As a simple example, we give the following result, which amounts to an improved version of [Proposition 2.2](#).

5.4. Theorem. *Let c and d be mutually generic Cohen reals over M and let $g: \omega \rightarrow 2$ be a function (not necessarily in M). Then $c \lambda_d g$ is a Cohen real over M .*

Proof. Let $D \in M$ be a dense open subset of $\text{Add}(\omega, 1)$. We wish to check that $c' := c \lambda_d g$ meets D . Consider the following subset of $\text{Add}(\omega, 2)$:

$$E := \{(p, q) \mid \text{for some } k < \omega, p \text{ is } (k, D)\text{-immune and } q \text{ has 0s on the interval } [k, |p|)\}$$

The set E is dense in $\text{Add}(\omega, 2)$. This is because, given any condition (p, q) in this poset (where we may assume that $|p| = |q|$), we can simply find a D -immunization p' of p , using [Proposition 5.3](#) and pad q with 0s up to the length of p' .

Since E is dense, there is a pair $(p, q) \in E$ such that p and q are initial segments of c and d , respectively. Now consider $p \lambda_q g$, where we identify q with $q^{-1}[\{1\}]$. This is an initial segment of $c \lambda_d g$, and it lies in D , since p was immune up to the last 1 of q below $|p|$. Therefore $c \lambda_d g$ meets D and is truly a Cohen real over M . \square

5.5. Corollary. *Let c be a Cohen real over M . Then there is another Cohen real d over M which does not amalgamate with c .*

Proof. Let d' be a Cohen real mutually generic with c , and fix a catastrophic real z for M . By [Theorem 5.4](#), the real $d := d' \lambda_c z$ is Cohen over M . But any model containing both c and d would be able to recover z as the bits of d on the coordinates in c , and therefore cannot be a forcing extension of M . Thus c and d are not amalgamable. \square

This corollary provides a glimpse of possible improvements to results establishing the existence of $*$ -embeddings into the generic multiverse, such as [Corollary 2.10](#). Specifically, it hints at a possible extension-of-embeddings phenomenon, whereby a $*$ -embedding of a subposet into the generic multiverse could be extended to a $*$ -embedding of the whole poset. One is reminded of the structure of the Turing degrees, which enjoys similar universality properties as we have shown for the generic multiverse. The fact that the Turing degrees do exhibit the extension-of-embeddings phenomenon, modulo an exact list of obstructions, was shown by Slaman and Soare [[SS01](#)], and it seems likely that a similar result can be proved for the generic multiverse.

5.6. Question. *Does the generic multiverse exhibit an extension-of-embeddings phenomenon?*

Before we perform surgery on a family of Cohen reals to obtain a desired pattern of amalgamability, we will have to prepare them slightly. We now describe this transformation of a family of reals, called *priming*.

5.7. Definition. Let $y: \omega \rightarrow 2$ be a real. We say that i is a *good point* for y if $y(i) = 1$ and $y(i+1) = 0$. We write $G(y) := \{i \mid i \text{ is a good point for } y\}$ and $C(y) := \{i+1 \mid i \in G(y)\}$.

If x, y, z are reals, we write $x[z/y] := x \lambda_{C(y)} z$. Often y will be clear from context and we will simply write $x[z]$.

If x_0, \dots, x_{n-1}, y are reals, the *primed* version of \vec{x} with respect to y is \vec{x}' , where

$$x'_k(i) := \begin{cases} 1 & i \in G(y) \\ 0 & i \in C(y) \\ 0 & i \notin G(y) \cup C(y) \text{ and } |G(y) \cap i| \equiv k \pmod{n} \\ x_k(i) & \text{otherwise} \end{cases}$$

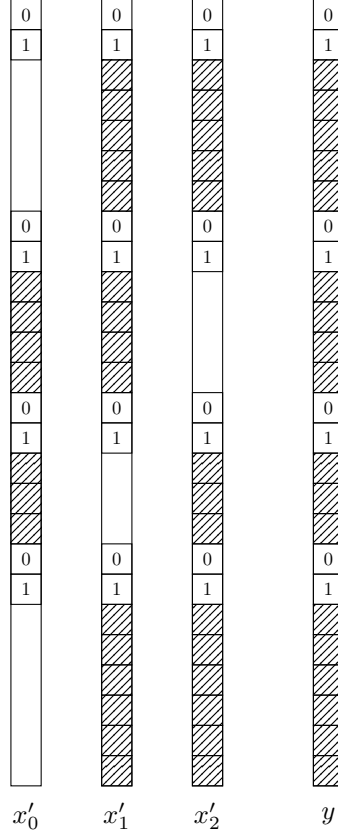
We say that k is *inactive* at i if $i \notin G(y) \cup C(y)$ and $|G(y) \cap i| \equiv k \pmod{n}$. \diamond

Given reals $x_0, z_0, \dots, x_{n-1}, z_{n-1}, y$ we will quite often abuse notation and write \vec{x}' for the tuple of the primed reals x'_0, \dots, x'_{n-1} , and $\vec{x}[\vec{z}]$ for the tuple of reals $x_0[z_0], \dots, x_{n-1}[z_{n-1}]$.

The diagram in [Figure 3](#) represents the result of priming three reals x_0, x_1, x_2 with respect to y . The blank blocks represent indices where a particular column is inactive and where 0s have been inserted. The operation $x'_0[z/y]$ then amounts to placing z onto the indicated bits of x'_0 that occur just above the all-one rows. The picture is again divided into blocks, fit between the good points of y . As we shall see, the structure of the blocks will ensure that genericity properties of the original reals are preserved after priming and surgery. Since the blocks are intended to accept modification via surgery, we might describe the whole construction as the mutable blockchain.

An easy but important observation to make is that, given reals \vec{x} and y , the primed versions \vec{x}' are never mutually generic Cohen reals, even if the original reals \vec{x} were. This is because we have ensured in the priming procedure that two consecutive all-one rows never appear, but this would have to happen in a mutually generic family. The following lemma shows that this is essentially the only failure of mutual genericity. If we start with mutually generic Cohen reals, then, after priming and possibly even additional surgery, any proper subfamily of these modified reals remains mutually generic. Moreover, this remains true even if we add some unprimed reals into the mix.

5.8. Lemma. *Suppose that $x_0, \dots, x_{n-1}, u, y$ are mutually generic Cohen reals. Let \vec{x}' be the primed version of \vec{x} with respect to y . Let $A \subset n$ and fix reals w and z_k for $k \in A$. Then $\{x'_k[z_k] \mid k \in A\} \cup \{u[w]\}$ are also mutually generic Cohen reals.*


 FIGURE 3. Priming x_0, x_1, x_2 with respect to y

Proof. Let D be a dense open subset of $\text{Add}(\omega, A + 1)$ and fix some $\ell \in n \setminus A$. Consider the following subset of $\text{Add}(\omega, n + 2)$:

$$E := \{(\vec{p}, q, r) \mid \exists m < \omega: (\vec{p} \upharpoonright_A, q) \text{ is } (m, D)\text{-immune}, \\ r \text{ has only 0s on the interval } [m, |(\vec{p} \upharpoonright_A, q)|], \text{ and} \\ \ell \text{ is inactive on } [m, |(\vec{p} \upharpoonright_A, q)|]\}$$

A similar argument to the one in the proof of [Theorem 5.4](#) shows that E is dense. Therefore (\vec{x}, u, y) meet E . As in that previous proof, it again follows by an immunity argument that $((\vec{x}' \upharpoonright_A)[\vec{z}], u[w])$ meet D . \square

Using essentially the same idea, we can extend the lemma to deal with a whole matrix of mutually generic Cohen reals, with each row being primed with respect to a different real. We omit the proof, since it is quite similar to the one just given.

5.9. Lemma. *Suppose that $x_0^j, \dots, x_{n_j-1}^j, u, y^j, v$ for $j < J$ are mutually generic Cohen reals. Let $(\vec{x}^j)'$ be the primed version of \vec{x}^j with respect to y^j . Let $A^j \subset n$ and fix reals w and z_k^j for $k \in A^j$. Then $\{(x_k^j)'[z_k^j/y^j] \mid k \in A^j\} \cup \{u[w/v]\}$ are also mutually generic Cohen reals.*

Before we give the main theorem of this section, let us present an easier version that we will use as a building block in the coming proof.

5.10. Theorem. *Let n be a natural number and let $A \subseteq n$. Let y be a Cohen real over M . Then there are amalgamable Cohen reals c_0, \dots, c_{n-1} over M and a real z such that the family $\{c_k[z/y] \mid k \in B\}$ is nonamalgamable if $A \subseteq B$ and mutually generic otherwise.*

Proof. Let x_0, \dots, x_{n-1} be mutually generic Cohen reals over $M[y]$ and let z be catastrophic for M . Let $\{c_k \mid k \in A\}$ be the primed versions of $\{x_k \mid k \in A\}$ with respect to y and let $c_k := x_k$ if

$k \notin A$. The reals $\{c_i \mid i < n\}$ are Cohen over M by [Lemma 5.8](#), and they are clearly amalgamable since they all live in $M[x_0, \dots, x_{n-1}, y]$. Note that, given the set $\{c_k[z/y] \mid k \in A\}$, it is simple to decode z : Find those i for which $c_k[z/y](i) = 1$ for all $k \in A$ and recover $C(y)$ from that information, which then give z . Therefore, the family $\{c_k[z/y] \mid k \in B\}$ is not amalgamable if $A \subseteq B$. On the other hand, if $A \not\subseteq B$, [Lemma 5.8](#) implies that $\{c_k[z/y] \mid k \in B\}$ are mutually generic Cohen reals. \square

5.11. Theorem. *Let n be a natural number. Then there are amalgamable Cohen reals c_0, \dots, c_{n-1} over M and another real y such that, for any $\mathcal{A} \subseteq \mathcal{P}(n)$ containing all singletons and closed under subsets, there is a real $z_{\mathcal{A}}$ such that $\{c_k \lambda_y z_{\mathcal{A}} \mid k \in A\}$ is amalgamable if and only if $A \in \mathcal{A}$.*

While this theorem gives a similar result as Mostowski's [Theorem 2.1](#), the conclusion is slightly stronger. In particular, we now obtain a fixed finite family of reals c_k which, by judicious use of surgery, can be modified to witness any desired pattern of amalgamability. Furthermore, as will be apparent from the proof, the real y can be chosen to be quite sparse (in terms of asymptotic density, say), so that the ‘‘postoperative’’ reals will resemble the ‘‘preoperative’’ ones as closely as desired.

Proof. Let $\{x_k^A, y^A \mid k < n, A \subseteq n, |A| \geq 2\}$ be mutually generic Cohen reals over M . Let $\{c_k^A \mid k \in A\}$ be the primed versions of $\{x_k^A \mid k \in A\}$ with respect to y^A and let $c_k^A := x_k^A$ for $k \notin A$. For each $k < n$ let $c_k := \bigoplus_A c_k^A$ be the join, obtained by interleaving bits for example; we also let $y = \bigoplus_A C(y^A)$. Essentially we prepared a matrix of Cohen reals, with one row for each subset $A \subseteq n$, and priming the reals in the A -th row with indices in A . We claim these c_k , the joins of the columns of this matrix, are as desired.

Let us first see that each c_k is Cohen over M . Let A_0, A_1, \dots, A_N be an enumeration of subsets of n of size at least 2. Recall that $\{x_k^A, y^A \mid A \subseteq n, |A| \geq 2\}$ are mutually generic over M . It then follows from [Lemma 5.8](#) that $c_k^{A_0}$ is Cohen over $M[x_k^{A_1}, y^{A_1}, \dots, x_k^{A_N}, y^{A_N}]$. Notice that $c_k^{A_j}$ is in this model for each $j > 0$. Similarly, $c_k^{A_1}$ is Cohen over $M[x_k^{A_2}, y^{A_2}, \dots, x_k^{A_N}, y^{A_N}]$, and this model contains all of the reals $c_k^{A_j}$ for $j > 1$. Repeating this step, we see that the reals $c_k^{A_0}, \dots, c_k^{A_N}$ are mutually generic over M . Therefore their join c_k is also Cohen over M . Furthermore, as we have said, the model $M[x_k^{A_j}, y^{A_j} \mid j \leq N]$ has all of the reals $c_k^{A_j}$, and therefore also c_k . Therefore all of the reals c_k appear in the model $M[x_k^A, y^A \mid k < n, A \subseteq n, |A| \geq 2]$, and are thus amalgamable.

Now fix a family \mathcal{A} as in the statement of the theorem. For each $A \in \mathcal{P}(n) \setminus \mathcal{A}$ there is, by [Theorem 5.10](#), a real z^A such that $\{c_k^A[z^A/y^A] \mid k \in B\}$ is nonamalgamable if $A \subseteq B$ and mutually generic otherwise. For $A \in \mathcal{A}$ let $z^A \equiv 0$. Now let $z_{\mathcal{A}} := \bigoplus_A z^A$, so that $c_k \lambda_y z_{\mathcal{A}} = \bigoplus_A c_k^A [z^A/y^A]$.

If $A \notin \mathcal{A}$ then $\{c_k \lambda_y z_{\mathcal{A}} \mid k \in A\}$ is not amalgamable, since from these reals we could compute $\{c_k^A [z^A/y^A] \mid k \in A\}$ and we know these reals are not amalgamable.

Now consider the situation when $A \in \mathcal{A}$. Let us write $M^A = M[x_l^D, y^D \mid D \in \mathcal{A}, l \in A]$. On the one hand, note that for any $B \in \mathcal{A}$ and any $k \in A$ we have $c_k^B [z^B/y^B] \in M^A$. On the other hand, applying [Lemma 5.9](#) over M^A shows that, since $B \setminus A \neq \emptyset$ for any $B \notin \mathcal{A}$, the reals in $\{c_k^B [z^B/y^B] \mid B \notin \mathcal{A}, k \in A\}$ are mutually generic over M^A . It follows that there is a single Cohen extension of M^A containing all the reals in $\{c_k^B [z^B/y^B] \mid B \subseteq n, k \in A\}$, and so the reals $c_k \lambda_y z_{\mathcal{A}}$ are amalgamable over M . \square

A simple extension of the preceding proof yields a sequence of Cohen reals c_0, c_1, \dots that uniformly realizes all finite patterns of amalgamability.

5.12. Theorem. *There are amalgamable Cohen reals c_0, c_1, \dots over M and a real y such that, for any natural number n and any $\mathcal{A} \subseteq \mathcal{P}(n)$ containing all singletons and closed under subsets, there is a real $z_{\mathcal{A}}$ such that, for any $A \subseteq n$, the family $\{c_k \lambda_y z_{\mathcal{A}} \mid k \in A\}$ is amalgamable if and only if $A \in \mathcal{A}$.*

It is unclear whether the mutable blockchain argument can be modified to achieve the kind of control over the intersections of the resulting generic extensions which would be required to obtain

a $*$ -embedding into the generic multiverse, or whether it can accommodate infinite patterns \mathcal{A} (compare with the modifications going from [Theorem 2.1](#) to [Theorem 2.6](#)).

6. MORE STRUCTURE IN THE COHEN MULTIVERSE

In this section we would like to further discuss the structure of the generic multiverse, but more specifically restricted to those extensions which may be obtained by adding a single Cohen real. [Theorem 2.9](#) and [Corollary 2.10](#) show that this structure exhibits a large degree of universality—any finite poset $*$ -embeds into it, along with certain infinite ones, and the embedding can also be made to preserve infima. We believe these results should be seen as analogous to the many theorems about the order structure of the Turing degrees (or even the c. e. degrees), which exhibits a similar kind of universality (although not quite in terms of $*$ -embeddings).

It turns out that there are nontrivial restrictions on the infinitary structure of the Cohen multiverse. The second author and Venturi [[Ham16](#), Theorem 13] showed that countable increasing chains in this structure have upper bounds (a similar but not completely analogous result holds for the whole generic multiverse; see the aforementioned paper and also [[FHR15](#)]).

6.1. Theorem (Hamkins–Venturi). *Let $\mathcal{C} = \{M[c_n] \mid n < \omega\}$ be a family of Cohen extensions of M such that every finite subfamily of \mathcal{C} has a least upper bound in the Cohen multiverse. Then \mathcal{C} is amalgamable in the Cohen multiverse.*

On the other hand, $*$ -embeddings, as we have defined them, do not capture this infinitary behaviour very well. For example, the reader will quickly convince herself that the inclusion of ω into $\omega + 1$ is a $*$ -embedding. Moreover, any ideal on a set X (such as the ideal of finite subsets) $*$ -embeds into the full power set $\mathcal{P}(X)$, regardless of which (infinitary) upper bounds exist in the ideal. It follows that any ideal on ω $*$ -embeds into the Cohen multiverse, even though the ideal will typically not have the finite-obstacle property that we required in our theorems.

To better explore this aspect of the structure of the Cohen multiverse, one might wish to work with a stronger form of $*$ -embedding.

6.2. Definition. Let $f: P \rightarrow Q$ be a $*$ -embedding between posets. We say that f is a *strong $*$ -embedding* if, whenever $X \subseteq P$ does not have an upper bound in P , its image $f[X]$ also does not have an upper bound in Q . \diamond

It is easy to see that if \mathcal{A} is a family of sets as in [Theorem 2.6](#) then any $*$ -embedding of \mathcal{A} already is a strong $*$ -embedding. Therefore the $*$ -embeddings obtained in [Theorem 2.9](#) and [Corollary 2.10](#) can be taken to be strong $*$ -embeddings. On the other hand, it follows from [Theorem 6.1](#) that the finite-obstacle property is a necessary condition for a family \mathcal{A} to strongly $*$ -embed into the Cohen multiverse in any way similar to what we have seen.

6.3. Proposition. *Let $I \in M$ and let $\mathcal{A} \in M$ be a family of subsets of I , closed under subsets in M . Suppose that \mathcal{A} strongly $*$ -embeds into the Cohen multiverse via a $*$ -embedding that preserves finite suprema. Then \mathcal{A} is defined by finite obstacles in M .*

Note that the $*$ -embeddings we constructed in [section 2](#) (and any embedding which, like those, is based on arrangements of product generics) were strong $*$ -embeddings and preserved finite suprema.

Proof. Let f be the strong $*$ -embedding of \mathcal{A} into the Cohen multiverse. Fix a set $B \in M$ all of whose finite subsets are in \mathcal{A} . We aim to show that B is also in \mathcal{A} . Since M is countable, there are, from the point of view of V , only countably many finite subsets of B . Let \mathcal{C} be the range of f restricted to the finite subsets of B . Note that \mathcal{A} is closed under unions of finite subsets of B , so, since f preserves suprema, \mathcal{C} is closed under suprema. The family \mathcal{C} is countable, since M is countable, therefore it follows from [Theorem 6.1](#) that \mathcal{C} is amalgamable in the Cohen multiverse. By the definition of a strong $*$ -embedding we can thus conclude that $B \in \mathcal{A}$. So any subset of I not in \mathcal{A} must contain a finite set not in \mathcal{A} , and thus \mathcal{A} is defined by finite obstacles. The set of obstacles can be taken as the \subseteq -minimal subsets of I not in \mathcal{A} , and this set is in M . \square

Continuing with the comparison of the Cohen multiverse and Turing degrees, we now turn to the existence of suprema and infima. First of all, it is an easy consequence of the intermediate model theorem that we can find a supremum of finitely many models in the generic multiverse as long as they are amalgamable.

6.4. Proposition. *A finite subset of the generic multiverse has a supremum if and only if it is amalgamable.*

Proof. Consider two models M_0 and M_1 in the generic multiverse. It follows from the DDG (see [section 1](#)) that both of these two models are forcing extensions of a common ground model. We may assume without loss of generality that this common ground model is just M and write $M_0 = M[G_0]$ and $M_1 = M[G_1]$ for some M -generic G_0 and G_1 . We may also assume that G_0 and G_1 are sets of ordinals. Now suppose that M_0 and M_1 are amalgamable with upper bound M_2 and consider the model $M(G_0, G_1) = \bigcup_{\alpha < \text{Ord}^M} L(V_\alpha^M, G_0, G_1)^{M_2}$. This is an inner model of M_2 and it satisfies choice since G_0 and G_1 are sets of ordinals and therefore easily well-orderable. It then follows from the intermediate model theorem that $M(G_0, G_1)$ is a forcing extension of M , and it is also clearly the least upper bound for $M[G_0]$ and $M[G_1]$. \square

In contrast, it is known that countable increasing sequences of Turing degrees never have a supremum and that not all pairs of Turing degrees have an infimum. These facts are usually established by showing the existence of *exact pairs* of Turing degrees, and the analogous argument will show that the same holds in the Cohen multiverse.

6.5. Definition. Let \mathcal{E} be a family of forcing extensions of M . We say that two forcing extensions $M[G]$ and $M[H]$ form an *exact pair* for \mathcal{E} if each of them is an upper bound for \mathcal{E} , and any model in the generic multiverse below both $M[G]$ and $M[H]$ is below some model in \mathcal{E} . \diamond

6.6. Theorem. *Suppose that $M[c_0] \subseteq M[c_1] \subseteq \dots$ is a countable tower of Cohen extensions of M . Then there are Cohen extensions $M[d_0]$ and $M[d_1]$ forming an exact pair for this tower.*

Proof. First we replace the Cohen reals c_n with a different family c'_n giving the same models, but with the additional property that any finitely many of the c'_n are mutually generic. We begin by letting $c'_0 = c_0$. By the intermediate model theorem, $M[c_1]$ is a forcing extension of $M[c_0]$ by a quotient of the forcing $\text{Add}(\omega, 1)$. But any such quotient is itself equivalent to $\text{Add}(\omega, 1)$ and so there is a Cohen real c'_1 over $M[c_0] = M[c'_0]$ such that $M[c_1] = M[c'_0, c'_1]$. Continuing in this way we can find all of the reals c'_n as required.

We will build the exact pair by filling in two ω -by- ω matrices with 0s and 1s in ω many steps. Throughout the construction we will maintain the requirement that, in each matrix, finitely many of the leftmost columns have been completely filled and the n -th such column differs at most finitely much from the corresponding real c'_n , and that beyond these columns the matrix is empty. In particular, at any stage of the construction, the matrices will essentially consist of finitely many mutually generic Cohen reals.

In our construction we will run through a fixed enumeration of all dense subsets of $\text{Add}(\omega, \omega)$, as well as all pairs of $\text{Add}(\omega, \omega)$ -names for reals in M . So suppose that we are at some stage of the construction and we are handed a dense set D and a pair of names σ and τ . Suppose that, in each matrix, the first N many columns have been filled. Since these columns are mutually generic, they meet the projection of D onto the first N many coordinates. It follows that we can extend each matrix by finitely many bits to meet the whole dense set D . Now consider the two names σ and τ , or rather, consider the two $\text{Add}(\omega, \omega \setminus N)$ -names σ' and τ' obtained by partially evaluating σ and τ by the first N columns of the respective matrices. Let p and q be the finite parts of the two matrices, seen as conditions in $\text{Add}(\omega, \omega \setminus N)$. If there is a number $k \in \omega$ such that there are extensions $p' \leq p$ and $q' \leq q$ which force $\check{k} \in \sigma'$ and $\check{k} \notin \tau'$ (or vice versa), respectively, we extend our matrices by these conditions; otherwise we do nothing. Note that if such a k does not exist then p and q already decide all formulas $\check{k} \in \sigma'$ and $\check{k} \in \tau'$ and decide them in the same way. Finally, we consider each partially filled column in our matrices and complete it to (almost) match the corresponding real c'_n . This finishes this step of the construction.

At the end of the construction let d_0 and d_1 be the reals represented by the two matrices we built. It is clear that both of these reals are Cohen over M , since we met all the relevant dense sets in M , and it is also clear that $M[d_0]$ and $M[d_1]$ are upper bounds for the tower we started with, since d_0 and d_1 code all the reals c'_n (up to finite modification). It thus only remains to show that no models can be fit between the tower and the new pair of extensions. By the intermediate model theorem, any putative such model would have to itself be a Cohen extension of M , so it suffices to show that any real common to both $M[d_0]$ and $M[d_1]$ already appears at some point in the tower. So let x be such a real and fix two names σ and τ from M such that $\sigma^{d_0} = \tau^{d_1} = x$. This pair of names was considered at some point in the construction, and we attempted at that time to ensure that σ^{d_0} and τ^{d_1} would differ at some k . By our hypothesis there was no k like this, so, as we argued above, the set $x = \sigma^{d_0} = \tau^{d_1}$ appeared already in the extension $M[c_N]$. \square

6.7. Corollary. *No countable increasing tower of Cohen extensions has a least upper bound in the generic multiverse.*

Proof. By the preceding theorem any such tower admits an exact pair $M[d_0], M[d_1]$, and by definition there is no upper bound for the tower that would lie below both of these two models. Therefore there can be no least upper bound for the tower. \square

A weaker version of [Theorem 6.6](#) was first obtained by Balcar and Hájek [BH78]; they proved that some tower of Cohen extensions admits an exact pair (see also the elaboration by Truss [Tru78]). Our results generalize this further to show that any countable tower of Cohen extensions admits an exact pair and, as we are about to see, that any upper bound for a tower may be extended to an exact pair.

6.8. Theorem. *Suppose $M[c_0] \subseteq M[c_1] \subseteq \dots$ is a countable tower of Cohen extensions of M with upper bound $M[d_0]$. Then there is a Cohen real d_1 over M such that $M[d_0]$ and $M[d_1]$ form an exact pair for this tower.*

Proof. We proceed as in the proof of [Theorem 6.6](#), replacing the reals c_n with the finitely mutually generic c'_n , and building the real d_1 as an ω -by- ω matrix whose n -th column is almost equal to c'_n . Arguing as before, at each step we first meet a dense subset of $\text{Add}(\omega, \omega)$ from M and then consider a pair of names σ and τ for reals. The only case in which we should act is if d_1 (or the part of it that we have constructed thus far) does not decide the value of τ . In that case there is some $k \in \omega$ such that our approximation of d_1 does not decide $k \in \tau$, and we extend it to make τ and σ^{d_0} differ at k (note that d_0 is fully generic, so σ^{d_0} is a fully fledged real). Afterwards we fill in the nonempty columns of d_1 using the given Cohen reals c'_n .

The proof is finished exactly the same way as before. The constructed real is clearly Cohen over M , and any real that can be written as $\sigma^{d_0} = \tau^{d_1}$ must have already been fully decided at some initial stage of the construction and must thus appear in some extension $M[c_N]$. \square

6.9. Corollary. *For any Cohen extension $M[c]$ there is another Cohen extension $M[d]$ such that these two models do not have a greatest lower bound in the generic multiverse.*

Proof. Let c be a Cohen real over M , seen as an ω -by- ω matrix, and let c_n be the real consisting of the first $(n+1)$ many columns of c . Then all of the reals c_n are Cohen over M and they form a tower $M \subseteq M[c_0] \subseteq M[c_1] \subseteq \dots$ with upper bound $M[c]$. By the preceding theorem there is a Cohen real d such that $M[c]$ and $M[d]$ form an exact pair for this tower. But these two models cannot have a greatest lower bound, since each model in the tower is a lower bound for them, but, by definition, no lower bound exists above the tower. \square

Note that if d_0 and d_1 are an exact pair for the tower $M[c_0] \subseteq M[c_1] \subseteq M[c_2] \subseteq \dots$, it will not in general be the case that $M[d_0] \cap M[d_1] = \bigcup_n M[c_n]$, even though this is true at the level of reals. For example $\{x \mid \exists n: x =^* c_n\}$, the set of all finite modifications of the c_n , is in both $M[d_0]$ and $M[d_1]$ as constructed in the proof of [Theorem 6.6](#), but is clearly not in any $M[c_n]$.

Finally, we wish to examine the relationship between the existence of suprema and infima of models in the multiverse and mutual genericity. Two mutually generic extensions clearly have a supremum

and, by a result of Solovay, an infimum. As we will see, neither of these two implications can be reversed and, in fact, the existence of a supremum does not imply the existence of an infimum or vice versa.

6.10. Theorem. *There is a pair of models in the Cohen multiverse that have an infimum but do not have a supremum, and another pair of models that have a supremum but do not have an infimum. In particular, neither the existence of a supremum nor the existence of an infimum imply that the two models are mutually generic (over the infimum, in the second case).*

Proof. For the first part, consider two extensions $M[c_0], M[c_1]$ given by [Theorem 2.6](#) for the family $\mathcal{A} = \{\{0\}, \{1\}\}$. In particular, $M[c_0]$ and $M[c_1]$ do not amalgamate, so they do not have a supremum, and their intersection is M , which implies that M is also their greatest lower bound in the multiverse.

For the second part we use a result due to Truss [[Tru78](#)] which states that in any Cohen extension $M[c]$ there are two Cohen reals d, e over M such that $M[d]$ and $M[e]$ do not have an infimum. His construction builds an exact pair over a tower much like we did in [Theorem 6.6](#), but instead of building the pair by induction and ensuring that the two generics almost agree on each column, Truss uses a generic sequence of finite modifications on the columns of c . In the end we are left with two models $M[d]$ and $M[e]$ with no infimum, but they are amalgamable and therefore have a supremum, by [Proposition 6.4](#). \square

We can strengthen the preceding result a bit to show that not even the existence of both a supremum and an infimum suffices for mutual genericity.

6.11. Theorem. *There are Cohen extensions $M[c]$ and $M[d]$ which have both a supremum and an infimum in the full generic multiverse, but which are not mutually generic over their infimum.*

Proof. Let $M[G]$ be an extension of M by the forcing to collapse $(2^\omega)^+$ to be countable. Inspecting the proof of [Theorem 2.6](#), it is clear that that whole construction may be carried out in $M[G]$ (with G taking the role of the catastrophic real) to produce Cohen reals $c, d \in M[G]$ over M such that any extension of M containing both c and d also contains G and that any real contained in $M[c] \cap M[d]$ is already contained in M .⁷

It follows that M is the infimum of $M[c]$ and $M[d]$. This is because any other lower bound above M would have to be a forcing extension of M by a subforcing of Cohen forcing, by the intermediate model theorem, and all such extensions are generated by a real, but we assumed that $M[c]$ and $M[d]$ only have the reals of M in common. On the other hand, $M[c]$ and $M[d]$ clearly have $M[G]$ as their supremum.

However, $M[c]$ and $M[d]$ cannot be mutually generic. If they were, then their supremum would also be a Cohen extension of M , which it clearly is not. \square

The Cohen extensions in the preceding theorem were amalgamable in the full multiverse but not in the Cohen multiverse. It is less clear whether the conclusion of the theorem still holds if we require the supremum and infimum to exist in the Cohen multiverse, but we expect that it does.

6.12. Question. *Let $M[c]$ and $M[d]$ be Cohen extensions of M with infimum M and a supremum in the Cohen multiverse. Are $M[c]$ and $M[d]$ mutually generic extensions of M ?*

Our construction of an exact pair of Cohen reals can be carried out in a sufficiently large collapse extension of M , and Truss showed that just a Cohen extension suffices. Both of these imply that exact pairs can be amalgamable. On the other hand, it is not clear whether the construction can be combined in some way with the blockchain construction to ensure nonamalgamability.

6.13. Question. *Can an exact pair be nonamalgamable?*

⁷ Some care is needed to ensure this last requirement. Instead of going through all pairs of Cohen names for subsets of M , as in the proof of [Theorem 2.6](#), we just go through pairs of names for reals, and there are countably many of those in $M[G]$.

Finally, acknowledging the utility of exact pairs of Cohen reals, we must ask whether their existence is a peculiar fact about Cohen forcing, or whether they can be constructed in other multiverses as well.

6.14. **Question.** *Do forcing notions beyond Cohen forcing also exhibit exact pairs?*

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