The strength of the class forcing theorem

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Set forcing

Theorem (Forcing theorem)

$\mathbb{P}$ is a set-sized separative partial order. $G \subseteq \mathbb{P}$ generic over V.

- *$V[G] \models \varphi(a_1, \ldots, a_n)$ iff $p \Vdash \varphi(\dot{a}_1, \ldots, \dot{a}_n)$ for some $p \in G$.*
- *$p \Vdash \varphi$ is definable. (For each k the relation $p \Vdash \varphi$ for Σ_k formulae φ is definable.)*

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This is a theorem of (a fragment of) ZFC.

Second-order set theory

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Definition

Gödel–Bernays set theory GBC has axioms

- ZFC for the first-order part;
- Extensionality for classes;
- Replacement: for class function F and set a we have $F''a$ is a set;
- Global Choice: there is a bijection $\mathrm{Ord} \to V$; and
- Elementary Comprehension: for φ with only set quantifiers and class A the following is a class:

$$\{x : \varphi(x, A)\}.$$

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Fact

GBC *is conservative over* ZFC*: for first-order* φ, GBC $\vdash \varphi$ *iff* ZFC $\vdash \varphi$.

A stronger second-order set theory

Definition

Kelley–Morse set theory KM has the axioms of axioms of GBC plus

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Fact

KM *is not conservative over* ZFC, *e.g. proving* Con(ZFC).

Pretame forcings

Theorem (Friedman)

GBC *proves that all pretame class forcing notions satisfy the forcing theorem.*

GBC doesn't prove the full class forcing theorem...

Theorem (Holy, Krapft, Lücke, Njegomir, Schlicht)

There is a (definable) class forcing notion $\mathbb{F}$ so that first-order truth is definable from $\Vdash_{\mathbb{F}}$ (for quantifier-free formulae).

Corollary

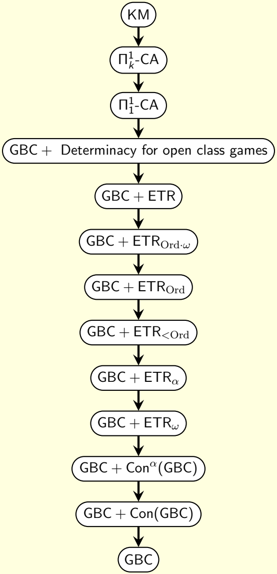
Over GBC, *the forcing theorem for $\mathbb{F}$ implies* Con(*ZFC*).

...But KM does

Theorem (Antos)

KM *proves the forcing theorem for all class forcing notions.*

What is the strength of the class forcing theorem?



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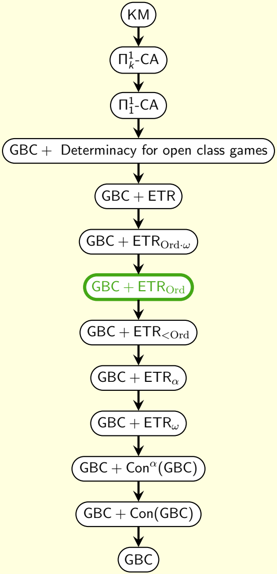

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$\mathbb{P}$ *admits a forcing relation for atomic formulae* if there are relations

$$p \Vdash \sigma \in \tau, \quad p \Vdash \sigma \subseteq \tau, \quad p \Vdash \sigma = \tau$$

satisfying

- $p \Vdash \sigma \in \tau$ iff there are densely many $q \leq p$ so that there is $\langle \rho, r \rangle \in \tau$ with $q \leq r$ and $q \Vdash \sigma = \rho$;
- $p \Vdash \sigma \subseteq \tau$ iff $\langle \rho, r \rangle \in \sigma$ and $q' \leq p, r$ implies there is $q \leq q'$ with $q \Vdash \rho \in \tau$; and
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- $p \Vdash \sigma = \tau$ iff $p \Vdash \sigma \subseteq \tau$ and $p \Vdash \tau \subseteq \sigma$.

- We can unify the three relations into a single relation, since they are distinguished syntactically.

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- We can unify the three relations into a single relation, since they are distinguished syntactically.
- $p \Vdash \sigma \subseteq \tau$ can be expressed in terms of $p \Vdash \sigma \in \tau$ and $p \Vdash \sigma = \tau$, so it's merely a convenience.
- Verifying that a class is $\Vdash$ is first-order (in the parameter $\mathbb{P}$).

What do we really mean by the class forcing theorem?

Definition

Φ a collection of first-order formulae, closed under subformulae. $\mathbb{P}$ *admits a forcing relation for* Φ if there is a relation $p \Vdash \varphi$ defined for $\varphi \in \Phi$ satisfying

- $\Vdash$ is defined on atomic formulae as before;
- For class name Σ, $p \Vdash \tau \in \Sigma$ iff there are densely many $q \leq p$ so that there is $\langle \rho, r \rangle \in \Sigma$ with $q \leq r$ and $q \Vdash \tau = \rho$;
- $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$;
- $p \Vdash \neg\varphi$ iff there is no $q \leq p$ so that $q \Vdash \varphi$; and
- $p \Vdash \forall x \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for all $\mathbb{P}$-names τ.

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- $p \Vdash \forall x \varphi(x)$ iff $p \Vdash \varphi(\tau)$ for all $\mathbb{P}$-names τ.

$\mathbb{P}$ *admits a forcing relation for a formula* φ if there is Φ containing all instances of $\varphi(\bar{\tau})$ so that $\mathbb{P}$ admits a forcing relation for Φ.

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Lemma Schema (GBC)

If $\mathbb{P}$ admits a forcing relation for atomic formulae then it admits a forcing relation for φ for any φ in the forcing language.

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Proof.

By induction in the meta-theory. □

Constructing actual forcing extensions

Suppose $\mathfrak{M} = (M, \mathcal{X}) \models \mathrm{GBC}$; $\mathbb{P} \in \mathcal{X}$ admits a forcing relation for all φ; $G \subseteq \mathbb{P}$ generic over $\mathfrak{M}$.

Constructing actual forcing extensions

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$$\begin{aligned} \sigma =_G \tau \quad & \text{iff} \quad \exists p \in G \ p \Vdash \sigma = \tau \\ \sigma \in_G \tau \quad & \text{iff} \quad \exists p \in G \ p \Vdash \sigma \in \tau \end{aligned}$$

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Then $=_G$ is an equivalence relation and a congruence with respect to $\in_G$. Set $\mathfrak{M}[G]$ to consist of the $=_G$-equivalence classes with $\in_G$ for its membership relation.

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Theorem

$\mathfrak{M}[G] \models \varphi([\tau_0], \ldots, [\tau_m])$ iff there is $p \in G$ so that $p \Vdash \varphi(\tau_0, \ldots, \tau_m)$.

Elementary transfinite recursion

Definition (Fujimoto)

Elementary transfinite recursion ETR is the schema asserting that for any well-order Γ and any first-order $\varphi(x, Y, A)$ (class parameter A) there is a class $S \subseteq \operatorname{dom} \Gamma \times V$ which is a solution of the recursion

$$S_a = \{x : \varphi(x, S \restriction a, A)\}$$

where $S_a = \{x : \langle a, x \rangle \in S\}$ and $S \restriction a = S \cap ((\Gamma \restriction a) \times V)$.

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Definition

$\mathsf{ETR}_{\mathrm{Ord}}$ is the restriction of ETR to recursions of height $\leq \mathrm{Ord}$.

The strength of ETR

Proposition

Over GBC, ETR *implies* Con(GBC).

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Proof.

The Tarskian definition of truth is an elementary recursion of height ω. □

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Theorem (Sato)

Π^1_1-CA *proves* Con(GBC + ETR)*.*

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Over GBC, ETR *implies* $\mathrm{Con}(\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}})$.

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Proof deferred to a later slide.

Getting forcing relations for atomic formulae

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Proof.

$\Vdash$ is defined via an elementary recursion. This is a recursion along $\in$ on $\mathbb{P}$-names. So we can organize it as a recursion of height Ord. $\square$

The uniform forcing relation

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Note that the uniform forcing relation cannot be definable from $\mathbb{P}$ for danger of contradicting Tarski's theorem on the undefinability of truth. In particular, we don't have uniform forcing relations for ordinary set forcing in ZFC.

Getting uniform forcing relations

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From before we have $\Vdash_{\mathbb{P}}$ for atomic formulae. Extending to all formulae is itself an elementary recursion. □

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From before we have $\Vdash_{\mathbb{P}}$ for atomic formulae. Extending to all formulae is itself an elementary recursion. □

Once we've seen that every forcing having a forcing relation for atomic formulae implies $\mathrm{ETR}_{\mathrm{Ord}}$ we will get:

Corollary (GBC)

If every class forcing admits a forcing relation for atomic formulae then every class forcing admits a uniform forcing relation.

Iterated truth

Definition

An Ord-*iterated truth predicate for first-order truth* is a class Tr consisting of triples $\langle \beta, \varphi, \vec{a} \rangle$, where $\beta \in \mathrm{Ord}$, φ is a first-order formula in $\mathcal{L}_{\mathsf{ZFC}}(\hat{\mathrm{Tr}})$, and $\vec{a}$ is a valuation for φ satisfying the following:

(a) Tr judges the truth of atomic statements correctly:

$$\mathrm{Tr}(\beta, x = y, \langle a, b \rangle) \quad \text{iff} \quad a = b$$

$$\mathrm{Tr}(\beta, x \in y, \langle a, b \rangle) \quad \text{iff} \quad a \in b$$

(b) Tr judges atomic assertions of the truth predicate self-coherently:

$$\mathrm{Tr}(\beta, \hat{\mathrm{Tr}}(x, y, z), \langle \alpha, \varphi, \vec{a} \rangle) \quad \text{iff} \quad \alpha < \beta \text{ and } \mathrm{Tr}(\alpha, \varphi, \vec{a})$$

(c) Tr performs Boolean logic correctly:

$$\mathrm{Tr}(\beta, \varphi \wedge \psi, \vec{a}) \quad \text{iff} \quad \mathrm{Tr}(\beta, \varphi, \vec{a}) \text{ and } \mathrm{Tr}(\beta, \psi, \vec{a})$$

$$\mathrm{Tr}(\beta, \neg\varphi, \vec{a}) \quad \text{iff} \quad \neg\mathrm{Tr}(\beta, \varphi, \vec{a})$$

(d) Tr performs quantifier logic correctly:

$$\mathrm{Tr}(\beta, \forall x\, \varphi, \vec{a}) \quad \text{iff} \quad \forall b\, \mathrm{Tr}(\beta, \varphi, b^\frown \vec{a})$$

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An Ord-*iterated truth predicate for first-order truth relative to a parameter* A is a class Tr consisting of triples $\langle \beta, \varphi, \vec{a} \rangle$, where $\beta \in \mathrm{Ord}$, φ is a first-order formula in $\mathcal{L}_{\mathsf{ZFC}}(\hat{\mathrm{Tr}}, \hat{A})$, and $\vec{a}$ is a valuation for φ satisfying the previous conditions plus:

(a') Tr judges the truth of atomic assertions about $\hat{A}$ correctly:

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(a') Tr judges the truth of atomic assertions about $\hat{A}$ correctly:

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- $\mathrm{Tr}_\Gamma(A)$ denotes *the* Γ-iterated truth predicate relative to A.
- Tr_Γ denotes *the* Γ-iterated truth predicate relative to no parameter.

ETR iff iterated truth

Theorem (Fujimoto)

Over GBC, ETR *is equivalent to* $\mathrm{Tr}_\Gamma(A)$ *exists for all well-orders* Γ *and all classes* A.

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Proof.

($\Rightarrow$) $\mathrm{Tr}_\Gamma(A)$ is defined via an elementary recursion of height $\omega \cdot \Gamma$.

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Proof.

($\Leftarrow$) Let $T = \mathrm{Tr}_\Gamma(A)$. Consider an instance of ETR, iterating $\varphi(x, S, A)$ along Γ. That is, we want to find $S \subseteq \operatorname{dom}\Gamma \times V$ so that $S_a = \{x : \varphi(x, S \restriction a, A)\}$ for all $a \in \operatorname{dom}\Gamma$.

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By the fixed-point lemma find $\bar{\varphi}$ so that $(V, \in, A, T \upharpoonright a) \models \bar{\varphi}(x, a)$ iff $(V, \in, A, S \upharpoonright a) \models \varphi(x, a)$.

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Then $S = \{\langle a, x\rangle : (a, \bar{\varphi}, x) \in T\}$ is as desired. $\square$

ETR iff iterated truth

Corollary

Over GBC, $\mathrm{ETR}_{\mathrm{Ord}}$ *is equivalent to* $\mathrm{Tr}_{\mathrm{Ord}}(A)$ *exists for all classes* A*.*

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Proof.

To prove $(\Rightarrow)$ before we used a recursion of height $\omega \cdot \Gamma$, but $\omega \cdot \mathrm{Ord} = \mathrm{Ord}$. So $\text{ETR}_{\mathrm{Ord}}$ suffices to construct Ord-iterated truth predicates. $(\Leftarrow)$ goes through the same. □

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Corollary

Let $\Gamma \geq \omega^\omega$*. Over* GBC, ETR_Γ *is equivalent to* $\mathrm{Tr}_\Gamma(A)$ *exists for all classes* A*.*

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Proof.

$\Gamma \geq \omega^\omega$ implies $\omega \cdot \Gamma < \Gamma + \Gamma$ and ETR_Γ is equivalent to $\text{ETR}_{\Gamma+\Gamma}$. □

Separating ETR and $\mathrm{ETR}_{\mathrm{Ord}}$

Theorem

Suppose $(M, \mathcal{X}) \models \mathsf{GBC} + \mathsf{ETR}$. *Then there is* $\mathcal{Y} \subseteq \mathcal{X}$ *coded in* $\mathcal{X}$ *so that* $(M, \mathcal{Y}) \models \mathsf{GBC} + \mathsf{ETR}_{\mathrm{Ord}}$.

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Proof.

Fix $G \in \mathcal{X}$ a global well-order. Define

$$\mathcal{Y} = \bigcup_{\Gamma < \mathrm{Ord}\cdot\omega} \mathsf{Def}(M, \mathrm{Tr}_\Gamma(G)).$$

Then $(M, \mathcal{Y}) \models \mathsf{GBC}$.

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Then $(M, \mathcal{Y}) \models \mathsf{GBC}$. It satisfies $\mathsf{ETR}_{\mathrm{Ord}}$ because if $A \in \mathsf{Def}(M, \mathrm{Tr}_\Gamma(G))$ for $\Gamma < \mathrm{Ord} \cdot \omega$ then $\mathrm{Tr}_{\mathrm{Ord}}(A)$ is in $\mathsf{Def}(M, \mathrm{Tr}_{\Gamma+\mathrm{Ord}+1}(G))$. □

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Corollary

Over GBC, ETR *implies* $\mathsf{Con}(\mathsf{GBC} + \mathsf{ETR}_{\mathrm{Ord}})$.

Separating levels of ETR

Theorem

Suppose $(M, \mathcal{X}) \models \mathsf{GBC} + \mathsf{ETR}_{\Gamma \cdot \omega}$ *for* $\Gamma \in \mathcal{X}$. *Then there is* $\mathcal{Y} \subseteq \mathcal{X}$ *coded in* $\mathcal{X}$ *so that* $(M, \mathcal{Y}) \models \mathsf{GBC} + \mathsf{ETR}_{\Gamma}$.

Outline of class forcing theorem $\Rightarrow$ ETR_{Ord}

Theorem

Over GBC, if every class forcing admits its forcing relation for atomic formulae then ETR_{Ord} *holds.*

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Infinitary languages

Definition

A a class. $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$ is the partial infinitary language relative to the parameter A. Formulae formed according to the following schema.

- Atomic formulae: $x = y$, $x \in y$, $x \in \hat{A}$;
- If φ is a formula then so is $\neg\varphi$;
- If φ_i are formulae for all $i \in I$ a set, so are $\bigvee_{i \in I} \varphi_i$ and $\bigwedge_{i \in I} \varphi_i$, so long as the φ_i have finitely many free free variables.
- If φ is a formula then so is $\exists x \varphi(x)$ and $\forall x \varphi(x)$.

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Definition

A a class. $\mathcal{L}_{\mathrm{Ord},0}(\in, \hat{A})$ is the the quantifier-free infinitary language relative to the parameter A. It is the restriction of $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$ to quantifier-free formulae.

Getting the uniform $\mathcal{L}_{\mathrm{Ord},0}(\in, V^{\mathbb{F}_A})$-forcing relation

Lemma (Holy, Krapft, Lücke, Njegomir, Schlicht)

If a class forcing notion $\mathbb{P}$ admits a forcing relation for atomic formulae then it admits a uniform forcing relation in the quantifier-free infinitary forcing language $\mathcal{L}_{\mathrm{Ord},0}(\in, V^{\mathbb{P}}, \dot{G})$.

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Key point: this is done via a purely syntactic translation, not making reference to generic filters or truth in a forcing extension.

Truth predicates for the infinitary language

Definition

A a class. An $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$-*truth predicate* is a class Tr consisting of pairs $\langle \varphi, \vec{a} \rangle$, where φ is an $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$-formula and $\vec{a}$ is a valuation for φ satisfying the following:

(a) Tr judges the truth of atomic statements correctly:

$$\begin{aligned} \mathrm{Tr}(x = y, \langle a, b \rangle) \quad &\text{iff} \quad a = b \\ \mathrm{Tr}(x \in y, \langle a, b \rangle) \quad &\text{iff} \quad a \in b \\ \mathrm{Tr}(x \in \hat{A}, \langle a \rangle) \quad &\text{iff} \quad a \in A \end{aligned}$$

(b) Tr performs Boolean logic correctly:

$$\begin{aligned} \mathrm{Tr}\left(\bigwedge_{i \in I} \varphi_i, \vec{a} \right) \quad &\text{iff} \quad \mathrm{Tr}\left(\varphi_i, \vec{a} \right) \text{ for all } i \in I \\ \mathrm{Tr}(\neg \varphi, \vec{a}) \quad &\text{iff} \quad \neg \mathrm{Tr}(\varphi, \vec{a}) \end{aligned}$$

(c) Tr performs quantifier logic correctly:

$$\mathrm{Tr}(\forall x\, \varphi, \vec{a}) \quad \text{iff} \quad \forall b\, \mathrm{Tr}(\varphi, b^\frown \vec{a})$$

Infinitary truth predicates $\to$ Ord-iterated truth predicates

Theorem

A a class. If there is an $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$-truth predicate then there is an Ord-iterated $\mathcal{L}_{\omega,\omega}(\in, \hat{A})$-truth predicate.

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Intuition.

Define a certain syntactic translation

$$(\beta, \varphi) \mapsto \varphi^*_\beta$$
$$\mathrm{Ord} \times \mathcal{L}_{\omega,\omega}(\in, \hat{A}) \to \mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$$

so that $\varphi(\vec{a})$ is true at level β iff $\varphi^*_\beta(\vec{a})$ is true.

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Key point: This translation is defined via a set-like recursion of height Ord, so it can be done just from GBC.

The $*$-translation (easy cases)

The translation is defined by induction on β and φ:

- Atomic formulae:

$$\begin{aligned} [x = y]^*_\beta &= [x = y] \\ [x \in y]^*_\beta &= [x \in y] \\ [x \in \hat{A}]^*_\beta &= [x \in \hat{A}] \end{aligned}$$

- Boolean combinations:

$$\begin{aligned} [\varphi \wedge \psi]^*_\beta &= [\varphi^*_\beta \wedge \psi^*_\beta] \\ [\neg \varphi]^*_\beta &= [\neg \varphi^*_\beta] \end{aligned}$$

- Quantifiers:

$$[\forall x \varphi]^*_\beta = [\forall x \varphi^*_\beta]$$

The $*$-translation (substantive case)

The translation is defined by induction on β and φ:

- $[\hat{\mathrm{Tr}}(x, y, z)]^*_\beta$ is the assertion that
  - x is some stage $\xi < \beta$;
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  Formally:

$$\bigvee_{\substack{\xi < \beta \\ \psi \in \mathcal{L}_{\omega,\omega}(\in, \hat{\mathrm{Tr}}, \hat{A})}} \left[\text{“}x = \xi\text{”} \wedge \text{“}y = \psi\text{”} \wedge \exists \vec{a}\ \mathrm{valuation}_\psi(z, \vec{a}) \wedge \psi^*_\xi(\vec{a}) \right]$$

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Let T be the $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$-truth predicate. Define the proposed Ord-iterated truth predicate Tr as $(\beta, \varphi, \vec{a}) \in \mathrm{Tr}$ iff $(\varphi^*_\beta, \vec{a}) \in \mathrm{T}$.

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Let T be the $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$-truth predicate. Define the proposed Ord-iterated truth predicate Tr as $(\beta, \varphi, \vec{a}) \in \mathrm{Tr}$ iff $(\varphi^*_\beta, \vec{a}) \in \mathrm{T}$. Inductively show that Tr really is an iterated truth predicate. The only substantive case is:

- $(\beta, \hat{\mathrm{Tr}}(x, y, z), \langle \alpha, \varphi, \vec{a} \rangle) \in \mathrm{Tr}$ iff $\alpha < \beta$ and $(\alpha, \varphi, \vec{a}) \in \mathrm{Tr}$ □

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$\mathbb{F}_A$ is defined by adding certain suprema to $\mathrm{Coll}(\omega, V)$:

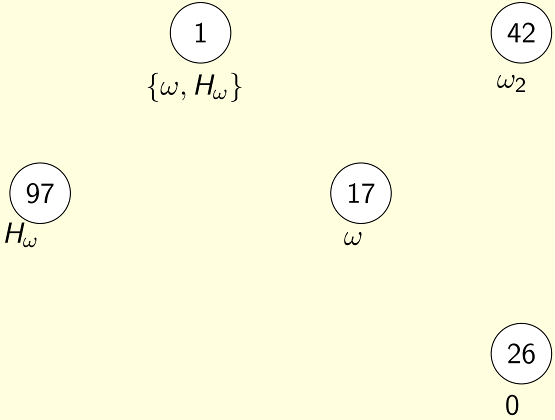
$$\mathbb{F}_A = \mathrm{Coll}(\omega, V) \sqcup \{e_{n,m} : n, m \in \omega\} \sqcup \{a_n : n \in \omega\}$$

where for $p \in \mathrm{Coll}(\omega, V)$

$$\begin{aligned} p \leq e_{n,m} \quad &\text{iff} \quad p(n) \in p(m) \\ p \leq a_n \quad &\text{iff} \quad p(n) \in A \end{aligned}$$

and $\mathbf{1}_{\mathbb{F}_A} = \emptyset \in \mathrm{Coll}(\omega, V)$ is above the new conditions.

A condition in the forcing $\mathbb{F}_{\mathrm{Ord}}$



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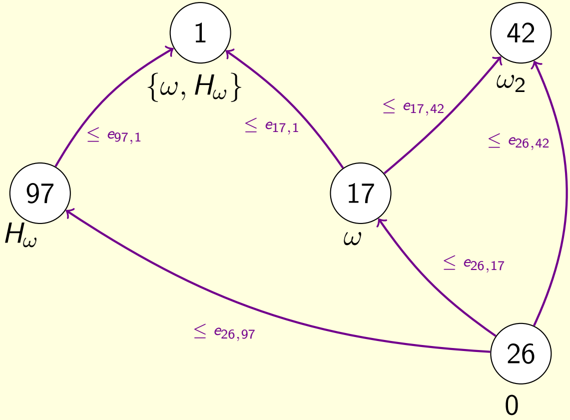

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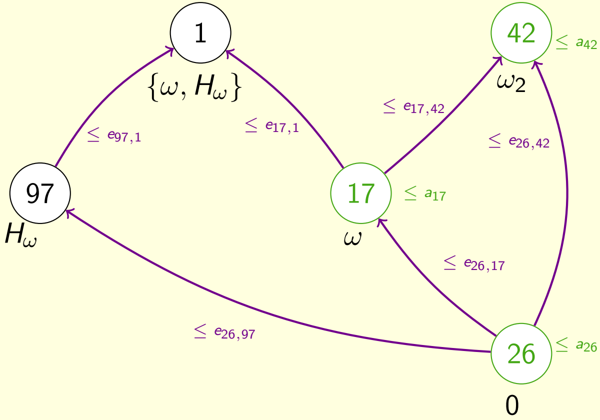

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These are set-sized names yet carry information about a proper class of conditions p.
For each set a define the name

$$\dot{n}_a = \{\langle \check{k}, \underbrace{\{\langle n, a\rangle\}}_{\in \mathrm{Coll}(\omega, V)} \rangle : k < n \in \omega\}.$$

$\dot{n}_a$ names the $n \in \omega$ that gets mapped to a by the generic bijection.

Defining truth from the forcing relation

Theorem

If $\mathbb{F}_A$ admits its uniform $\mathcal{L}_{\mathrm{Ord},0}(\in, V^{\mathbb{F}_A})$-forcing relation then the $\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A})$-truth predicate exists.

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Intuition.

Define a syntactic translation

$$\varphi \mapsto \varphi^\star$$
$$\mathcal{L}_{\mathrm{Ord},\omega}(\in, \hat{A}) \to \mathcal{L}_{\mathrm{Ord},0}(\in, V^{\mathbb{F}_A})$$

so that for $G \subseteq \mathbb{F}_A$ generic

$$(V, \in, A) \models \varphi(a) \quad \text{iff} \quad V[G] \models \left[(\omega, \dot{\varepsilon}^G, \dot{\mathrm{A}}^G) \models \varphi^\star((\dot{n}_a)^G)\right].$$

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The $\star$-translation

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- Atomic formulae:

$$\begin{aligned} [x = y]^\star &= [x = y] \\ [x \in y]^\star &= [x \,\dot{\varepsilon}\, y] \\ [x \in \hat{A}]^\star &= [x \in \dot{\mathrm{A}}] \end{aligned}$$

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Suppose the uniform $\mathcal{L}_{\mathrm{Ord},0}(\in, V^{\mathbb{F}_A})$-forcing relation exists. Define a class Tr as

$$(\varphi, \vec{a}) \in \mathrm{Tr} \quad \text{iff} \quad \mathbf{1} \Vdash_{\mathbb{F}_A} \varphi^\star(\dot{n}_{a_0}, \ldots, \dot{n}_{a_k}).$$

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So $\mathrm{ETR}_{\mathrm{Ord}}$ holds. □

Other equivalences

Theorem

The following are equivalent over GBC.

- *The class forcing theorem: all class forcing notions admit a truth predicate for atomic formulae.*
- *All class forcing notions admit a uniform* $\mathcal{L}_{\omega,\omega}(\in, V^{\mathbb{P}})$*-forcing relation.*
- *All class forcing notions admit a uniform* $\mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}(\in, V^{\mathbb{P}})$*-forcing relation.*
- $\mathsf{ETR}_{\mathrm{Ord}}$.
- Ord*-iterated* $\mathcal{L}_{\omega,\omega}(\in, \hat{A})$*-truth predicates exist.*
- $\mathcal{L}_{\mathrm{Ord},\omega}(\in, A)$*-truth predicates exist.*
- $\mathcal{L}_{\mathrm{Ord},\mathrm{Ord}}(\in, A)$*-truth predicates exist.*
- *Clopen class games of rank at most* $\mathrm{Ord}+1$ *are determined.*

Thank you!