Varities of class theoretic potentialism

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(This talk is about joint work with Carolin Antos and Neil Barton.)

A puzzle from the early days of set theory

Hochverehrter Freund.

Wie ich Ihnen vor einer Woche schrieb, liegt mir viel daran, Ihr Urtheil in gewissen fundamentalen Puncten der Mengenlehre zu erfahren und bitte ich Sie, die Ihnen dadurch verursachte Mühe mir zu verzeihen.

Gehen wir von dem Begriff einer bestimmten Vielheit (eines Systems, eines Inbegriffs) von Dingen aus, so hat sich mir die Nothwendigkeit herausgestellt, zweierlei Vielheiten (ich meine immer *bestimmte* Vielheiten) zu unterscheiden.

Eine Vielheit kann nämlich so beschaffen sein, daß die Annahme eines „Zusammenseins“ *aller* ihrer Elemente auf einen Widerspruch führt, so daß es unmöglich ist, die Vielheit als eine Einheit, als „ein fertiges Ding“ aufzufassen. Solche Vielheiten nenne ich *absolut unendliche* oder *inconsistente Vielheiten*.

Wie man sich leicht überzeugt, ist z. B. der „Inbegriff alles Denkbaren“ eine solche Vielheit; später werden sich noch andere Beispiele darbieten. [Anm. 1]

⋮

A. *Das System Ω aller Zahlen ist eine inconsistente, eine absolut unendliche Vielheit.*

Cantor in an 1899 letter to Dedekind

(Quoted from *Georg Cantor: Briefe*, pp. 408–409, eds H. Meschkowski & W. Nilson. 1991.)

A problem for the philosophy of set theory

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Problem

Give an explication of what classes are. In particular, answer how they differ from sets and why and in what sense a mathematician, if they accept sets, should also accept classes.

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- I'll then propose one possible species of answer, namely that of a potentialist view of classes.
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- I'll give some background, including some technical background, about potentialism, especially in how this viewpoint has previously been applied to first-order set theory, i.e. without looking at classes.
- I'll then investigate class potentialism, looking at some ways one might approach this and what the relative merits are.

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- Under this approach, the only classes that "exist" are the definable classes $\{x : \varphi(x)\}$, possibly allowing parameters in the definition.
- This approach explains how classes differ from sets, and it explains in what sense a mathematician who accepts sets should also accept classes.
- And it's known that a lot of talk that *prima facie* looks to be about classes can be interpreted in this framework: inner models, elementary embeddings, etc.

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But I claim there is a trouble with this approach.

An aside

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- A specific example: HOD is the inner model of hereditarily ordinal definable sets. That is, $x \in \mathrm{HOD}$ if $x \subseteq \mathrm{HOD}$ and x is definable from ordinal parameters.
- This definition may seem to need a truth predicate, but we can express it in a first-order way by merely asking that x be definable from ordinal parameters in some large enough V_α.

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- For many large cardinals, their most useful characterization is as the critical points of elementary embeddings.
- For example: κ is a measurable cardinal if it is the critical point of some elementary embedding $j : \mathrm{V} \to M$ into some inner model; that is, κ is the smallest ordinal so that $\kappa \neq j(\kappa)$.

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- These embeddings are proper classes, and saying an embedding is elementary seems to reference a truth predicate for V.
- But for these large cardinals, the embeddings are controlled by certain sets.
- For example: the embedding for a measurable cardinal can be given by a ultrapower of V by a measure on κ.
- So this is expresssible in ZFC.

The trouble with classes as mere syntax

- The problem is simple: while definable classes suffice for a lot of work in set theory, there is work where the previous sort of tricks to avoid undefinable classes do not apply and so set theorists really do need undefinable classes.
- So any approach which only admits definable classes is inadequate.

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Let's see some examples.

Kunen's inconsistency theorem

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If the only classes are the definable classes, this is a triviality:

- If j is definable without parameters, then so is the critical point of j. But any elementary embedding $\mathrm{V} \to \mathrm{V}$ must fix every definable object, so $j(\operatorname{crit} j) = \operatorname{crit} j$. $\lightning$
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- (A small extra argument then yields that we also cannot have such j definable with parameters.)

If we think, as set theorists as a whole do, that there is substantive content to Kunen's theorem, it is in showing such j cannot even be an undefinable class.

(This is a point raised by Hamkins, Kirmayer, and Perlmutter, and further developed by Roberts.)

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- (Gitman, Hamkins, Holy, Schlicht, W.) Indeed, we can exactly characterize a principle of class theory which is equivalent to the forcing theorem for every class forcing, namely the principle of Elementary Transfinite Recursion for recursions of height $\leq \mathrm{Ord}$. In particular, the class forcing theorem is equivalent to a principle asserting the existence of certain kinds of truth predicates.

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So if we want to be able to talk about class forcing in full generality we need undefinable classes.

The inner model hypothesis

The inner model hypothesis (IMH) is a sort of width reflection principle.

- (Friedman) The IMH asserts that if φ (no parameters) is true in an inner model of an outer model of V, then it is true in an inner model of V.
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- Question: How can we formalize the IMH, given that it's talking about arbitrary outer models, putting us well beyond V?
- (Antos, Barton, Friedman) The IMH can be formalized in class theory, via the use of the class Hyp(V).
- This puts things in the realm of impredicative comprehension, well beyond having just the definable classes.

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- But it implies that the only classes are the definable classes.
- This is a problem, as there is work within set theory that relies upon undefinable classes.
- If we are to give an account of classes that adequately founds work by set theorists, then we need a different account of what classes are.

Some approaches in the literature

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- This is not an exhaustive survey, and I'm focusing on approaches which have a certain commonality.

Fujimoto's liberal predicativism

Developed by Fujimoto, following earlier work by Parsons.

Quote (Fujimoto 2019)

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- Classes are distinct from sets because they are part of language—predicates—unlike sets.
- But this goes beyond just definable classes. In particular, Fujimoto explicitly allows truth predicates as admissible predicates.
- Indeed, he explicitly motivates his project with the need to allow talk of undefinable classes.

Linnebo's individuation of properties

Linnebo proposes a theory of properties on which properties are successively individuated along the ordinals.

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- Linnebo's properties are not classes, since they are intensional objects. But we can interpret class theory by looking at extensions of properties.
- There is a hierarchy of classes, based on 'when' a property giving that class is first individuated.

Fine's procedural postulationism

Fine proposes that we gain knowledge of mathematical objects by postulating their existence, where for him postulation amounts to more than mere assertion of truth.

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- Fine is concerned with a general mathematical context, but one could apply his ideas specifically to the class theoretic context, giving certain construction rules that govern the postulation of classes.
- Fine uses a modal framework here.

What's common in these approaches

These three approaches share a common feature.

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- Consider e.g. Fujimoto: If A is a class, then the truth predicate for the structure $(\mathrm{V}, \in, A)$ is also a class.
- This motivates a potentialist approach. We cannot have access to all classes at once, and there is always the potential to expand to allow new classes.

Potentialism about classes

The framework we wish to consider is as follows.

- We will assume actualism about sets: there is a determinate universe of sets which we can quantify over and cannot extend by adding new sets.

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- We will assume actualism about sets: there is a determinate universe of sets which we can quantify over and cannot extend by adding new sets.
    - Because sets and classes are different sorts of objects, one may naturally be led to this distinction.
    - Without commiting to any particular view of sets, this assumption focuses attention on what is properly class theoretic.
    - It would complicate the analysis to consider both sets and classes as potentially given.
    - In any case, the set theorist who takes sets potentially must hold the same for classes, so there's no need to convince them to adopt this framework.
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- We consider a transitive, reflexive extension relation between the structures.
  - This can just be the substructure relation $\subseteq$, but we can also refine to a more restrictive extension relation.
- We call this a potentialist system.
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- We can apply modal logic to understand a potentialist system. (More on this later.)

Set theoretic potentialism

This framework has previously been applied in (first-order) set theory, talking about just sets and not classes. Let's look at a few examples.

Zermelo

Zermelo articulated a vision of what set theory is based on an upward growing sequence of larger and larger domains—what we now call V_κ for the inaccessible cardinals κ.

Quote (Zermelo 1930)

This categorically determined domain of sets can then again be supplemented so as to become a normal domain of higher characteristic. Thus to every categorically determined totality of "boundary numbers" there follows a greater one, and the sequence of "all" boundary numbers is as unlimited as the number series itself, allowing for the possibility of associating to every transfinite index a particular boundary number in one-to-one fashion.

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This can naturally be understood in a potentialist framework by considering the potentialist system consisting of V_κ for κ inaccessible, working from a suitable background theory.

Forcing

- Given a universe of sets, we can extend to a larger universe of sets via Cohen's method of forcing.
- The potentialist perspective has been applied here (Hamkins, Löwe).
- Additionally, maximality principles for forcing can be naturally expressed in this modal context:

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- Two ways we could formalize this:
  - Look just at what can be expressed about forcing relations, without ever talking about an actual extension.
  - Fix a countable model M and look at the potentialist system consisting of forcing extensions of M. (The reason to require M to be countable is to ensure that, in the background universe, we have generics over M.)

Second-order arithmetic

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- Let me mention one specific potentialist system one could study in this context, so I can use it later as an example:
- The worlds are models $(\mathbb{N}, \mathcal{X})$ of second-order arithmetic with full impredicative comprehension, with $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ countable. Extension is just the substructure relation.

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- The worlds are models $(\mathbb{N}, \mathcal{X})$ of second-order arithmetic with full impredicative comprehension, with $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ countable. Extension is just the substructure relation.
- Two reasons to restrict to countable models:
  - If we allow $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ of arbitrary cardinality, then $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ would itself be a world. So this wouldn't accurately capture the idea of a domain we can always extend.
  - Much like in the forcing case, our models being countable allows us fuller tools to investigate the model theory of these structures.

The modal logic of a potentialist system

Given a potentialist system there's a natural interpretation of modal logic.

- For a world M:
- $M \models \Box\varphi$ if every extension of M satisfies φ;
- $M \models \Diamond\varphi$ if some extension of M satisfies φ.

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- $M \models \Box\varphi$ if every extension of M satisfies φ;
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A question:

- What are the modal validities of a potentialist system?
- That is, what are the modal propositional formulae which are true in any world under any substitution of formulae in the object language for the propositional variables?

Modal validities

Proposition

S4 *is valid for any potentialist system.*

$$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$$

$$\neg \Diamond p \Leftrightarrow \Box \neg p$$

$$\Box p \Rightarrow p$$

$$\Box p \Rightarrow \Box\Box p$$

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$$\Box p \Rightarrow p$$
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The first two axioms are valid for any Kripke frame, the third axiom is valid when the frame is reflexive, and the last is valid when the frame is transitive.

Modal validities

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- (Hamkins, Linnebo) Zermelian potentialism has S4.3 as its modal validities.

$$(\Diamond p \wedge \Diamond q) \Rightarrow [(p \wedge \Diamond q) \vee (\Diamond p \wedge q)]$$

  - This expresses a linearity to the truths.
  - Indeed, this potentialist system is linear as a partial order.

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- (Hamkins, Löwe) Forcing potentialism has S4.2 as its modal validities.

$$\Diamond\Box p \Rightarrow \Box\Diamond p$$

  - This expresses a directedness to the truths.
  - However, a theorem of Mostowski shows that it is not directed as a partial order.

Modal validities

The modal validities of a potentialist system give a qualitative measure of how truth behaves.

- Consider again the potentialist system consisting of countable ω-models of second-order arithmetic with full impredicative comprehension, ordered by extension.
- (Hamkins, W.) This potentialist system has precisely S4 as its modal validities.
  - This expresses an essential branching nature to the truths.
  - Indeed, the proof that S4 is an upper bound for the modal validities goes through a universal finite sequence, a definition for a finite sequence which you can freely extend by moving to the right larger world.

Potentialism about classes, formalized

Here is a general formal framework.

- Our background theory is ZFC plus suitable consistency assumptions.
- Consider a fixed countable transitive $M \models \mathrm{ZFC}$, and consider a fixed theory T for classes.
  Two possibilities:
  - Consider the potentialist system consisting of *all* $(M, \mathcal{X}) \models T$ with $\mathcal{X} \subseteq \mathcal{P}(M)$ countable, ordered by extension.
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  - Restrict this to a smaller collection of worlds.
- The choice of M and T can determine properties of this potentialist system. Part of this project is to understand just what the effect is.
- In particular, a bad choice of M for strong enough T can mean that the potentialist system has no worlds at all! We will tacitly assume we are not in this case.
  - (Krajewski) If $(M, \mathcal{X}) \models \mathrm{NBG}$ and $\mathcal{X}$ contains a truth predicate for M, then M must have unboundedly many worldly cardinals.

Truth and class theoretic potentialism

Truth predicates are an important example of undefinable classes.

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- They play an important role in some of the philosophical motivation, e.g. Fujimoto's liberal predicativism.

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Let me introduce some notation.

- Tr denotes the truth predicate for $(\mathrm{V}, \in)$, the (unique) class satisfying the Tarskian recursion.
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- $\mathrm{Tr}(A)$ denotes the truth predicate relative to the parameter A, i.e. the truth predicate for $(\mathrm{V}, \in, A)$.
- We can have $\mathrm{Tr}(\mathrm{Tr})$, $\mathrm{Tr}(\mathrm{Tr}(\mathrm{Tr}))$, and so on, giving a hierarchy of stronger and stronger languages. Such a sequence can be formulated as a single class, an iterated truth predicate, possibly of transfinite length.

  Let $\mathrm{Tr}_\Gamma(A)$ denote the (unique) length Γ iterated truth predicate relative to the parameter A, where Γ is a class well-order.

Minimal potentialist systems over M

Here are some minimal desiderata for a potentialist system.

2. Any world satisfies NBG, class theory with predicative comprehension. In particular, any world contains every definable class.
3. If A is a class in a world, there is a larger world with $\mathrm{Tr}(A)$ as a class.

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- This can be extended by allowing iterated truth predicates of transfinite length, looking at the worlds $\mathcal{X}_\alpha = \mathrm{Def}(M, \mathrm{Tr}_\alpha)$ for all α below some limit ordinal γ.
- If $\alpha + \beta < \gamma$ for all $\alpha < \gamma$ and $\beta < \omega^2$, then Leibman's technology of long ratchets lets us see that S4.3 is also an upper bound.

Building upward further to Fujimoto

- Fujimoto puts forth that NBG + ETR is the correct theory of classes. ETR is the principle of Elementary Transfinite Recursion, which can be equivalently formulated as asserting that $\mathrm{Tr}_\Gamma(A)$ exists for all Γ and A.
- We can alternatively cast Fujimoto's system in potentialist terms:
  1. $(M, \mathrm{Def}(M))$ is a world.
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- Under suitable assumptions on M, there is a smallest such potentialist system over M, and it will validate exactly S4.3.

  This is just a reformulation of the theorem (W.) that if M admits an expansion to a β-model of $\mathsf{NBG} + \mathsf{ETR}$ then it admits a smallest expansion.

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So far, no substantive new light has been shed. But this finer-grained look will be helpful when we ask for more.

Global choice and global well-orders

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- This can equivalently be formulated as the existence of a global well-order.
- Here are three ways we might have a global well-order:
  1. We might be lucky and have that M has a definable global well-order. This happens precisely when $M \models \mathrm{V} = \mathrm{HOD}$.
  2. We just assume that we start with some (possibly undefinable) global well-order given to us.
  3. We add a global well-order by class forcing, without adding any new sets. One way to do this is to add a Cohen-generic subclass to Ord. (This is forcing equivalent to the forcing to directly add a global well-order.)

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- The first option puts substantial constraints on the sets, so let's investigate the latter two options.

Starting with a global well-order

Let's modify our desiderata by requiring that $\mathrm{Def}(M, \triangleleft)$, rather than $\mathrm{Def}(M)$, be a world, where $\triangleleft$ is some global well-order.

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- If we're so lucky that $\lhd$ is definable, then this is the potentialist system already described, which validates exactly S4.3.
- If $\lhd$ comes from forcing over an appropriate model, it can define two Cohen generics $C, D \subseteq \mathrm{Ord}$ so that:
  1. $\mathrm{Tr}_i(C)$ and $\mathrm{Tr}_j(D)$ are not definable from Tr_k;
  2. $\mathrm{Tr}_i(C)$ is not definable from $\mathrm{Tr}_j(D)$, and vice versa.

  This then gives a failure of the .3 axiom: let φ assert that $\mathrm{Tr}_7(C)$ exists but $\mathrm{Tr}_7(D)$ does not, and let ψ assert that $\mathrm{Tr}_7(D)$ exists but $\mathrm{Tr}_7(C)$ does not. Then φ and ψ are both possible in the world $(M, \text{Def}(M, \lhd))$, but if φ is true then ψ is impossible and vice versa.

Starting with a global well-order

The lesson here is that if a global well-order is just handed to you randomly, then this can affect how your potentialist system behaves.

- The same construction yields that for appropriate M, the potentialist system consisting of all NBG-expansions for M will not validate the .3 axiom.

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Maybe things will be nicer if we don't start with a global well-order but add it in ourselves.

Adding a global well-order by class forcing

Let's keep the original three desiderata for our potentialist system, and add in a fourth one:

4. If $(M, \mathcal{X})$ is a world then so is $(M, \mathcal{X}[C])$ where C is a Cohen generic subclass of Ord. As remarked earlier, C will add a generic global well-order.

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- The point is, from $\mathrm{Tr}(A)$ we can define an Ord-sequence of dense subclasses so that meeting all of them guarantees genericity over $(M, \mathsf{Def}(M))$. We can meet these in minimal length and thereby code a real witnessing that M is countable, so that if we had both the generic C and $\mathrm{Tr}(A)$ we could recover this catastrophic real.

Adding a global well-order by class forcing

The lesson here is that if you grab a random generic global well-order then it can be quite destructive to your process of individuating more and more classes.

- If we started with a global well-order given to us and it happened to one of these destructive ones, that would destroy any hope of building up to even have truth predicates of length 1.
- The potentialist system consisting of all NBG-expansions for M has badly behaved worlds that seal off any possibility of extending further to world satisfying a stronger theory.

To sum up

At minimum, we want a few things from class potentialism.

- We want truth predicates to exist, possibly in an extension.
- We want access to a global well-order, possibly in an extension.

To have both of these we either have to

- Put restrictions on the sets, e.g. require they satisfy $\mathrm{V} = \mathrm{HOD}$;
- Put further restrictions on the worlds in the potentialist system; or
- Accept that there is some indeterminacy based on properties of a global well-order.

A more restrictive class potentialism

The construction of the destructive generics is blocked if we require all worlds to be closed under truth predicates; and a generalization of the construction for iterated truth predicates is blocked if we require all worlds to be closed under iterated truth predicates. This leads to the following desideratum, overwriting the previous three desiderata:

0. Every world satisfies NBG + ETR.

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One quick remark:

- Under certain assumptions on M, we have a smallest world. So this is compatible with an approach with a starting point—or more generally, we might want the order on the worlds to be well-founded. See Linnebo's account of properties, this also seems to be implicit in Fujimoto's liberal predicativism.

Stronger class theoretic principles

- Most of the analysis has focused on varities of truth predicates, which are relatively weak in the hierarchy of class theories.
- This is enough for many of the uses of definable classes.
- However, it is not enough for everything, for example formalizing the inner model hypothesis.

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I want to briefly discuss this stronger realm, and gesture a bit about how we might address it.

Stronger class theoretic principles

MK, class theory with full impredicative comprehension, has the following metatheorem:

- For each finite subtheory T of MK, MK proves that for every class A there is a coded V-submodel of T which contains A.
- A coded V-submodel is $(M, \mathcal{X})$ so that there a class X of pairs (i, x) where $\mathcal{X}$ consists of the slices $(X)_i = \{x : (i, x) \in X\}$ of X.
- Compare to the notion of coded ω-submodels from the study of second-order arithmetic.

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- Compare to the notion of coded ω-submodels from the study of second-order arithmetic.

A truth predicate $\mathrm{Tr}(A)$ is, up to recoding, a code for the V-submodel $(M, \mathrm{Def}(M, A))$, so this can be thought of as a generalization of the notion of truth predicates.

- Much like a truth predicate transcends a first-order language to allow more expressive power, a coded V-submodel gives a predicate which transcends a second-order language.

Stronger class theoretic principles

One example mentioned to motivate the need to give an account of classes which allows undefinable classes was the Antos, Barton, and Friedman work to formalize the inner model hypothesis. This requires much more than just truth predicates.

- Their work needed the class $\mathrm{Hyp}(\mathrm{V})$, adapting a concept from Barwise's admissible set theory to the class theoretic context.
- The existence of $\mathrm{Hyp}(\mathrm{V})$ implies the existence of a coded V-submodel of $\mathrm{NBG} + \mathrm{ETR}$.
- More generally, if $\mathrm{Hyp}(A)$ exists for every class A then every class is in a coded V-submodel of $\mathrm{NBG} + \mathrm{ETR}$.

Class theoretic potentialism with coded V-submodels

This suggests a new principle for class potentialism:

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However, some of the constructions with truth predicates generalize to this context.

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However, some of the constructions with truth predicates generalize to this context.

- For example, the destructive generic argument gives generics which cannot coexist with having $(M, \mathcal{X})$ as a coded V-submodel.

Perhaps the lesson here is that the above principle is too strong: we don't want to ask to be able to transcend all of $\mathcal{X}$ at once, but rather just ask to locally transcend $\mathcal{X}$.

A summary

- We want an account of classes which allows undefinable classes, as there are important uses of such in set theory.
- Class theoretic potentialism gives a framework in which many accounts of classes can be analyzed.
- In particular, it allows us to mathematize some of the questions, enabling the use of model theoretic tools.
- This helps to shed light on some nuances, similar to the application of the potentialist framework to first-order set theory.
- It possibly points toward a path to accommodate uses of class theory which require some amount of impredicative comprehension.

Danke schön!

Some references

- Kit Fine, "Relatively unrestricted quantification" 2006.
- Kentaro Fujimoto, "Predicativism about classes" 2019.
- Øystein Linnebo, "Sets, properties, and Unrestricted Quantification" 2006.
- Ernst Zermelo, "On boundary numbers and domains of sets: New investigations in the foundations of set theory" 1930.