Interpretations and bi-interpretations in second-order arithmetic

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they/them

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- (Mostowski) For each finite $T \subseteq \mathrm{PA}$, PA proves $\mathrm{Con}(T)$;
- (Visser) If T_0, T_1 are extensions of PA, then T_0 and T_1 are bi-interpretable iff they have the same deductive closure.

Interpretations

$\mathcal{M}$ and $\mathcal{N}$ are structures, possibly with different signatures.

- An interpretation of $\mathcal{N}$ in $\mathcal{M}$ is a definable copy of $\mathcal{N}$ in $\mathcal{M}$.
  - Definable $N^{\mathcal{I}} \subseteq M^k$ is the domain;
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- **Example:** The complex field $\mathbb{C}$ is interpreted in the real field $\mathbb{R}$ via the usual ordered pair idea.
- **Example:** ZFC can be interpreted in ZF via the constructible universe.
- **Example** There are other ways to interpret, e.g. via the class of hereditarily definable sets.

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- Analogy to another category: For topological spaces, it could be X embeds as a subspace of Y which embeds as a subspace of X, but X and Y are not homeomorphic: e.g. open vs closed intervals.
- We'll see soon that mutual interpretability does not imply bi-interpretability.

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Contrast to conservativity

Another notion of two theories having the same content is conservativity.

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But bi-interpretability is stronger than conservativity:

- **Example:** ACA_0 is not bi-interpretable with PA.
  (Because ACA_0 is finitely axiomatizable but PA is not.)

Lesson: ACA_0 and PA have the same arithmetical content, but ACA_0 has extra content beyond that.

(A fun exercise for the bored listener: come up with an explicit example of this extra content.)

Back to Visser's theorem

Definition: A theory T is tight if any two extensions in the same language are bi-interpretable if and only if they are deductively equivalent.

- Intuitively, this represents a sort of semantic completeness of T.

(Without the same language restriction this is trivial: e.g. PA + "the new unary predicate is the evens" versus PA + "the new unary predicate is the odds".)

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Examples:

- (Visser) PA
- (Enayat) ZF
- (Enayat) Z_2
- (Enayat) KM

- Z_2 is second-order arithmetic with full comprehension
- KM is Kelley–Morse class theory with full comprehension

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  - Interpret $\mathsf{ZFC} + \neg\mathsf{CH}$ in $\mathsf{ZFC} + \mathsf{CH}$ via the boolean ultrapower approach to forcing.

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  - Interpret $\mathsf{ZFC} + \neg\mathsf{CH}$ in $\mathsf{ZFC} + \mathsf{CH}$ via the boolean ultrapower approach to forcing.
- These interpretations lose information.
  - Dropping down to the constructible universe kills large cardinals, and you can't get them back by forcing.
  - Boolean ultrapowers give you a genuine interpretation, without needing a generic filter from outside the universe, but they destroy well-foundedness.

Enayat's theorem implies there are no interpretations you could choose to avoid this information loss.

Our main question

- Enayat extended Visser's ideas to apply to other important foundational theories.
- But the proofs use the full strength of these theories.
- These theories have natural hierarchies of increasingly stronger fragments.

$$\mathsf{I}\Sigma_0 \subseteq \mathsf{I}\Sigma_1 \subseteq \cdots \subseteq \mathsf{I}\Sigma_k \subseteq \cdots \subseteq \mathsf{PA}$$

$$\mathsf{ACA}_0 \subseteq \Pi^1_1\text{-}\mathsf{CA}_0 \subseteq \cdots \subseteq \Pi^1_k\text{-}\mathsf{CA}_0 \subseteq \cdots \subseteq \mathsf{Z}_2$$

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- Do we need the full strength of the theory to get tightness? Or are these fragments also tight?

Signs point to yes: Freire and I looked at fragments of $\mathsf{Z_2}$ and KM. Independently, Enayat has a different argument that overlaps with some of the cases we get.

Outline of the rest of the talk

- I'll give a special case of ACA being non-tight in some detail.
- I'll sketch how to get a uniform version of the bi-interpretation, to get ACA isn't tight.
- I'll then gesture toward how you can use the same core idea to show that the Π^1_k-CA are also non-tight.

The special case

I will demonstrate two models of ACA which satisfy different theories but are bi-interpretable.

Specifically, the minimum ω-model of ACA is bi-interpretable with a carefully chosen extension by Cohen forcing, and these models are distinguishable by their theories.

The minimum ω-model of ACA

- A model of second-order arithmetic is of the form $(M, \mathcal{X})$ where M are the numbers of the model and $\mathcal{X} \subseteq \mathcal{P}(M)$ are the sets.
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- ω-models satisfy full induction.
- Any ω-model of ACA_0 is a model of ACA.

- ACA is axiomatized by:
  - the axioms of discretely ordered semirings;
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- If $M \cong \omega$ then we call it an ω-model.
- ω-models satisfy full induction.
- Any ω-model of ACA_0 is a model of ACA.
- It's easy to see that the minimum ω-model of ACA is $(\omega, \mathcal{D})$, the finite ordinals equipped with their arithmetically definable subsets.

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Proof:

- For each $k \in \omega$, the restriction T_k of T to Σ_k formulae is in $\mathcal{D}$.
- So we can define that $\varphi[a]$ is in T iff there exists k so that there exists a set satisfying the definition of T_k which judges $\varphi[a]$ to be true.
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- This gives a Σ^1_1 definition of T.
- There's also Π^1_1 definition—any set that looks like a T_k which has $\varphi[a]$ in its domain judges $\varphi[a]$ to be true.
- Observe that both definitions can be carried out over any ω-model of ACA.
- Since this is Δ^1_1 it is absolute. All ω-models of ACA define T the same.

Identifying the minimum ω-model of ACA, and codes for higher order sets

Because T is definable, so is the property "$X \in \mathcal{D}$":

- $X \in \mathcal{D}$ iff there is $\varphi[a, x]$ so that $X = \{x : \varphi[a, x] \in \mathrm{T}\}$.

So "every set is arithmetically definable" is a single second-order assertion, and the only ω-model of ACA which satisfies it is the minimum ω-model.

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$\mathcal{D}$ is a set of sets of integers, but it can be coded by a single set of integers. The elements of $\mathcal{D}$ are the slices of T.

Because ω has a canonical well-order, we have a canonical enumeration of the element of $\mathcal{D}$: order them by the order of their smallest index in T.

Relativizing truth and definability

Consider $C \subseteq \omega$.

- $\mathrm{T}(C)$ is the truth predicate with C as a predicate;
- $\mathcal{D}(C)$ is the sets arithmetically definable from C.

The facts about T and $\mathcal{D}$ generalize to give:

- If $\mathcal{X}$ is an ω-model of ACA with $C \in \mathcal{X}$ then $\mathrm{T}(C)$ is definable over $\mathcal{X}$ and so is the predicate "$X \in \mathcal{D}(C)$".

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But if C is definable over $\mathcal{D}$ and generic over $\mathcal{D}$ for forcing then the truth lemma implies $\mathrm{T}(C)$ is definable over $\mathcal{D}$.

- An arithmetical formula $\varphi(C)$ is true iff there is $p \in C$ such that $p \Vdash \varphi(\dot{C})$.
- So we can define $\mathrm{T}(C)$ over $\mathcal{D}$ as: $\varphi[x, C] \in \mathrm{T}(C)$ iff there is $p \in C$ which forces $\varphi(x, \dot{C})$.

Defining a Cohen generic

Recall:

- Cohen forcing $\mathbb{P} = \mathrm{Add}(\omega, 1)$ is the infinite binary tree.
- A filter $C \subseteq \mathbb{P}$ is generic over $\mathcal{D}$ if it meets every dense subset of $\mathbb{P}$ from $\mathcal{D}$.

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From T we have a canonical enumeration of the ω many dense subsets. Now follow the usual proof of the Rasiowa–Sikorski lemma:

- Start with $p_0 = \emptyset$;
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Because $\mathcal{D}$ is uniformly definable over any ω-model of ACA, any ω-model of ACA defines C the same.

Putting it all together

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Theorem (Freire-W., independently Enayat)

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$(\omega, \mathcal{D})$ and $(\omega, \mathcal{U})$ are bi-interpretable but satisfy different extensions of ACA.

That $(\omega, \mathcal{U}) \models$ ACA is because forcing preserves arithmetical comprehension. And it satisfies "there is a set which is not arithmetically definable" whereas $(\omega, \mathcal{D})$ satisfies "every set is arithmetically definable".

Finally, use that $\mathrm{T}(C)$ is definable over $\mathcal{D}$ to build the two sides of the bi-interpretation.

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- But if two M-models have the same Σ_k-satisfaction classes, then they define T the same. For example, this happens if one is a forcing extension of the other.

Observation: Any model of ACA has a minimum ω-submodel (= submodel that agrees on ω) of ACA.

Non-tightness of ACA

Do the same definitions and arguments, but more carefully to check everything can be formalized, and you're done.

Let $D = \text{ACA} +$ "every set is arithmetical" and $U = \text{ACA} +$ "the canonical Cohen generic C exists and every set is arithmetical in C".

Theorem (Freire-W., independently Enayat)

The theories D and U are bi-interpretable. Consequently, ACA *is not tight.*

Consequently every subtheory of ACA, such as ACA_0 is also not tight.

From ACA to Π^1_k-CA

Abstractly, the strategy to prove the non-tightness of ACA was this:

- There is a minimum model of ACA.
- There is a second-order axiom to characterize this minimum model.
- We can define a canonical Cohen generic over this minimum model, and thereby get a definable choice for an extension of the minimum model.
- The minimum model and its canonical extension are bi-interpretable.
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To prove the non-tightness of Π^1_k-CA we follow the same strategy.

The main difficulty is, how do you definably characterize the minimum model of Π^1_k-CA? And how do you do it so that the same construction can be carried out with nonstandard models?

From ACA to Π^1_k-CA

Abstractly, the strategy to prove the non-tightness of ACA was this:

- There is a minimum model of ACA.
- There is a second-order axiom to characterize this minimum model.
- We can define a canonical Cohen generic over this minimum model, and thereby get a definable choice for an extension of the minimum model.
- The minimum model and its canonical extension are bi-interpretable.
- The construction machinery for the bi-interpretation works even over ω-nonstandard models.

To prove the non-tightness of Π^1_k-CA we follow the same strategy.

The main difficulty is, how do you definably characterize the minimum model of Π^1_k-CA? And how do you do it so that the same construction can be carried out with nonstandard models?

I'll sketch the highlights.

Second-order arithmetic is set theory in disguise, and minimum models

Strong subsystems of Z_2 are bi-interpretable with fragments of $ZFC^- +$ "every set is countable". (The minus in ZFC^- means *minus Powerset*).

The set theory $\to$ arithmetic direction is simple—restrict to subsets of ω.

The arithmetic $\to$ set theory direction is based on the idea, going back to Aczel and Scott, of coding sets as trees and constructing an appropriate membership relation between trees. A key observation, due to Simpson, is that ATR_0 suffices to carry out this interpretation.

Second-order arithmetic is set theory in disguise, and minimum models

Strong subsystems of Z_2 are bi-interpretable with fragments of $\mathrm{ZFC}^- +$ "every set is countable". (The minus in ZFC^- means *minus Powerset*).

- β-models of arithmetic are bi-interpretable with transitive models of set theory.

- An ω-model of arithmetic is a β-model if it is correct about which of its relations are well-founded.
- A model $\mathcal{M}$ of set theory is transitive if its membership relation is the true $\in$ and $\mathcal{M}$ is closed under $\in$:

$$x \in y \in M \text{ implies } x \in M.$$

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- β-models of arithmetic are bi-interpretable with transitive models of set theory.
- Levels of the constructible universe L give minimum transitive models of set theory, whence we get minimum β-models of arithmetic.

Important point! L has a definable global well-order, allowing us to make canonical choices.

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- β-models of arithmetic are bi-interpretable with transitive models of set theory.
- Levels of the constructible universe L give minimum transitive models of set theory, whence we get minimum β-models of arithmetic.
- We can definably characterize these minimum models.
- This works for ill-founded models, and is absolute to outer models ($=$ models with the same ordinals).

Key point: These levels of L don't satisfy Replacement, so they have definable cofinal maps.

We need a little fine structure theory to get a uniform definition.

Do the same argument

Once you know how to characterize minimum β-models you do the same construction.

- $(\omega, \mathcal{D}_k)$ = the minimum β-model of Π^1_k-CA.
- Over $(\omega, \mathcal{D}_k)$ define a canonical Cohen generic C.
- Then $(\omega, \mathcal{D}_k)$ and $(\omega, \mathcal{D}_k[C])$ are bi-interpretable and satisfy different theories.
- You can do this construction uniformly.

Π^1_k-CA is axiomatized by

- ACA; and
- Comprehension for Π^1_k-formulae.

Compare to Π^1_k-CA_0 which doesn't have full induction.

Theorem (Freire-W.)

For each finite k, Π^1_k-CA is not tight.

Related results

Freire and I were originally interested in the case of class theory, and only realized our constructions could be ported to arithmetic after the fact.

Theorem (Freire-W.)

The theories GB *and* $\mathsf{GB} + \Pi^1_k$-CA *are not tight.*

Related results

Freire and I were originally interested in the case of class theory, and only realized our constructions could be ported to arithmetic after the fact.

Theorem (Freire-W.)

The theories GB *and* $\mathrm{GB} + \Pi^1_k$-CA *are not tight.*

Independently to us, Ali Enayat proved:

Theorem (Enayat)

No finitely axiomatized subtheory of PA, ZF, Z_2, *or* KM *is tight.*

What remains to be done?

Conjecture (Enayat)

If T is a strict subtheory of Z_2 (or PA or ...) then T is not tight.

We know this in the cases:

- (Enayat) T is finitely axiomatizable;
- (Freire-W.) T has any amount of the Induction schema but only a bounded fragment of the Comprehension schema.

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An interesting open case:

- $T \subseteq \mathrm{ZF}$ has the full $\in$-Induction and Separation schemata but only a bounded fragment of Collection.

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Other uses of bi-interpretations with minimum models, e.g. in second-order arithmetic or higher recursion theory?

Thank you!

- Alfredo Roque Freire and Kameryn J. Williams,
  Non-tightness in class theory and second-order arithmetic.
  To appear: The Journal of Symbolic Logic.
  Pre-print: arXiv:2212.04445 [math.LO]