MATH455 HOMEWORK 10
DUE FRIDAY, APRIL 24

These exercises outline an alternate proof of Gödel's beta lemma. Do Exercises 1 and your choice of one of Exercises 2, 3, or 4.

Exercise 1. Show that "x is a power of 2" is expressible over $\mathcal{N}$ as a Δ_0 formula. [Hint: first show that "y divides x" and "2 divides y are expressible as Δ_0 formulae.]

Each natural number n can be represented by its *dyadic numeral*, the word $\varepsilon_{k-1} \cdots \varepsilon_1 \varepsilon_0$ where: k is a natural number (possibly $k = 0$, in which case the word is the empty word), each ε_i is 1 or 2, and $n = \sum_{i=0}^{k-1} \varepsilon_i 2^i$. This representation gives a bijection between the natural numbers and the set of words in the alphabet $\{1, 2\}$.

Exercise 2. Show that "the dyadic numeral representing y consists of all 2s" and "the dyadic numberals representing x and y have the same length" are both expressible over $\mathcal{N}$ as Δ_0 formulae. [Hint: by Exercise 1 you can expressed "z is a power of 2" in a Δ_0 way. How do powers of 2 relate to dyadic numerals?]

Exercise 3. Show that "the dyadic numeral for z is the concatenation of the dyadic numerals for x and y" and "the dyadic numeral for x is an initial segment of the dyadic numeral for z" are both expressible over $\mathcal{N}$ as Δ_0 formuale.

Let "$z = x^\frown y$" abbreviate "the dyadic numeral for z is the concatenation of the dyadic numerals for x and y" and $x \subseteq z$ abbreviate "the dyadic numeral for x is an initial segment of the dyadic numeral for z".

Consider the following two formulae:

- $\psi(x, y)$ holds iff $\exists u, w \leq y$ so that
  - $u \neq 1$, $u \subseteq y$ and u's dyadic numeral is a string of all 2s; and
  - $\forall z \leq y$ if $z \subseteq y$ and z's dyadic numeral is a string of all 2s then $z \subseteq u$; and
  - $w = u^\frown 1^\frown x^\frown 1^\frown u$ and $w \subseteq y$ and $u \not\subseteq x$.
- $\theta(x, y, z)$ holds iff either
  - $\psi(\pi(z, y), x)$ and $\forall v < z\ \neg\psi(\pi(v, y), x)$; or
  - $z = 0$ and $\forall u \leq x\ \neg\psi(\pi(u, y), x)$.[1]

By inspection plus the previous exercises, ψ and θ are both expreessible as Δ_0 formulae.

Exercise 4. Show that the formula $\theta(x, y, z)$ satisfies the conclusion to Gödel's beta lemma. That is, show that $\mathcal{N} \models \forall x \forall y \exists! z\ \theta(x, y, z)$ and for all sequences $\langle s_0, \ldots, s_{\ell-1} \rangle$ of natural numbers there is $s \in \mathcal{N}$ so that for all $i < \ell$ we have $\mathcal{N} \models \theta(s, i, s_i)$.

[1]Recall that $\pi : \mathbb{N}^2 \to \mathbb{N}$ is Cantor's pairing function, which is Δ_0 definable.