MATH455 HOMEWORK 4
DUE FRIDAY, FEBRUARY 21

Exercise 1. Suppose that a theory T has arbitrarily large finite models. Show that T has an infinite model.

Exercise 2. A linear order $(I, <)$ is called a *well-order* if there is no infinite descending sequence $a_0 > a_1 > \cdots > a_k > \cdots$, $k \in \mathbb{N}$, of elements from I.[1]

Show that if $\mathcal{I} = (I, <)$ is an infinite linear order then there is $\mathcal{J} \succ \mathcal{I}$ which is *not* a well-order. In particular, this shows that you cannot write down axioms in first-order logic which characterize being a well-order.

Exercise 3. Let $(I, <)$ be a linear order and suppose that $\langle \mathcal{M}_i : i \in I \rangle$ is a sequence of $\mathcal{L}$-structures so that if $i < j$ are elements of I then $\mathcal{M}_i \prec \mathcal{M}_j$. We call $\langle \mathcal{M}_i : i \in I \rangle$ an *elementary chain*. Show that if $M = \bigcup_{i \in I} M_i$ then there is a structure $\mathcal{M}$ with underlying set M so that $\mathcal{M}_i \prec \mathcal{M}$ for all $i \in I$. [Hint: first you have to say what the constants, functions, and relations of $\mathcal{M}$ are.]

Exercise 4. Let $\mathcal{M}$ be an infinite structure. Suppose $p(x) = \{\varphi(x)\}$ is a set of formulae in the language of $\mathcal{M}$ who all have x as their only free variable. Suppose that for all finite $p_0(x) \subseteq p(x)$ there is $a \in \mathcal{M}$ so that $\mathcal{M} \models \varphi[a]$ for all $\varphi(x) \in p_0(x)$. Show that there is $\mathcal{N} \supseteq \mathcal{M}$ so that there is $a \in \mathcal{N}$ with $\mathcal{N} \models \varphi[a]$ for all $\varphi(x) \in p(x)$.

[1] For example, $(\mathbb{N}, <)$ is a well-order but $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$ are not well-orders.