MATH455 HOMEWORK 6
DUE FRIDAY, MARCH 6

A general hint for this homework set: by the completeness + soundness theorems showing that Γ is consistent is equivalent to showing that Γ is satisfiable and $\Gamma \vdash \varphi$ is equivalent to $\Gamma \models \varphi$.

Exercise 1. Do exerercise 2.5 from the textbook page 82. (Recall the book's notation: S is a set of symbols and T^S is the set of terms using the symbols from S.)

Exercise 2. Let Γ be an inconsistent set of $\mathcal{L}$-formulae. Define, as we did in class, a structure $\mathcal{M}$ out of $\mathcal{L}$-terms:

- $t \sim t'$ if $\Gamma \vdash t = t'$ and $[t]$ is the $\sim$-equivalence class for t.
- $M = \{[t] : t \text{ is an } \mathcal{L}\text{-term}\}$.
- $c^{\mathcal{M}} = [c]$.
- $f^{\mathcal{M}}([t_1], \dots [t_k]) = [f(t_1, \dots, t_k)]$.
- $R^{\mathcal{M}}([t_1], \dots, [t_k])$ iff $\Gamma \vdash R(t_1, \dots, t_k)$.

Show that $\mathcal{M}$ is well-defined. Explicitly determine M. To what extent does this depend on the choice of inconsistent Γ.

Let Γ be a set of $\mathcal{L}$-formulae and $\Gamma^* \supseteq \Gamma$ be a set of $\mathcal{L}^*$-formulae, where $\mathcal{L}^* \supseteq \mathcal{L}$. Say that Γ^* is a *conservative extension* of Γ if given any $\mathcal{L}$-formula φ we have $\Gamma^* \vdash \varphi$ iff $\Gamma \vdash \varphi$. (That is, a conservative extension is one which doesn't prove any new formulae in the old language.)

Exercise 3. Fix a set I, a language $\mathcal{L}$, and a consistent set Γ of $\mathcal{L}$-formulae. Suppose that any structure which satisfies Γ is infinite. Extend $\mathcal{L}$ to $\mathcal{L}^*$ by adding constants c_i for each $i \in I$ and extend Γ to Γ^* by adding new axioms "$c_i \neq c_j$" for $i \neq j \in I$. Show that Γ^* is a conservative extension of Γ.

Exercise 4. Suppose Γ is a consistent and complete set of $\mathcal{L}$-formulae. Extend $\mathcal{L}$ to a new language $\mathcal{L}^*$ by adding constants $c_{\exists x\varphi}$ for each $\mathcal{L}^*$-formula $\exists x\varphi$, where the constant $c_{\exists x\varphi}$ does not occur in φ.[1] Extend Γ to Γ^* by adding new axioms "$\exists x\varphi \Rightarrow \varphi\frac{c_{\exists x\varphi}}{x}$". Show that Γ^* is a conservative extension of Γ.

[1] As in class, this seemingly circular definition is actually an inductive definition.