End-extensions of models of set theory and the Σ_1 universal finite sequence

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Models of Peano Arithmetic Seminar, CUNY
2020 July 29

Joint work with Joel David Hamkins

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Outline

1. Background.
2. The main theorem: the Σ_1 universal finite sequence.
3. Applications of the Σ_1 universal finite sequence.

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2. The main theorem: the Σ_1 universal finite sequence.
   This is where all the technical content is.
3. Applications of the Σ_1 universal finite sequence.

Woodin's universal algorithm

Theorem (Woodin)

There is a Turing machine program p with the following properties.

1. PA *proves that p enumerates a finite sequence.*
2. *Running p in $\mathbb{N}$ enumerates the empty sequence.*
3. *For any finite sequence s of natural numbers there is a nonstandard model of arithmetic M so that running p in M enumerates exactly s.*
4. *Moreover, if p enumerates s in M and $t \in M$ is a finite sequence extending s then there is an end-extension N of M in which running p enumerates exactly t.*

The actual infinite versus the potential infinite

- Two views on the infinite: actualist versus potentialist.
- The actualist believes in the existence of a completed infinite.
- The potentialist believes merely that there is an unbounded process. One can always add more but at no point does one get a completed infinite whole.
- Can we formalize these two views, say in the context of integer arithmetic?

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- The potentialist believes merely that there is an unbounded process. One can always add more but at no point does one get a completed infinite whole.
- Can we formalize these two views, say in the context of integer arithmetic?
- The actualist view: study the structure $\langle \mathbb{N}; 0, 1, +, \cdot, < \rangle$.
- Formalizing the potentialist view takes more technology.

Formalizing the potentialist view

- Formalize the actualist view: study the structure $\langle \mathbb{N}; 0, 1, +, \cdot, < \rangle$.
- (Linnebo–Shapiro) Formalize the potentialist view: study the collection of initial segments of $\mathbb{N}$, call them $\mathbb{N}_k$ for $k \in \mathbb{N}$, with the restrictions of the operations, and where this collection is ordered by extension.
- Can use modal logic to talk about truth in this potentialist system: $M \models \Diamond\varphi$ if φ holds in some extension of M and $M \models \Box\varphi$ if φ holds in all extensions of M.

Formalizing the potentialist view

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- Can use modal logic to talk about truth in this potentialist system: $M \models \Diamond \varphi$ if φ holds in some extension of M and $M \models \Box \varphi$ if φ holds in all extensions of M.
- (Linnebo–Shapiro) There is a correspondence from truth in the actualist system to modal truth in the potentialist system, call it $\varphi \mapsto \varphi^*$. Namely:

$$\mathbb{N} \models \forall x\ \varphi(x) \quad \text{iff} \quad \mathbb{N}_k \models \Box \forall x\ \varphi^*(x)$$
$$\mathbb{N} \models \exists x\ \varphi(x) \quad \text{iff} \quad \mathbb{N}_k \models \Diamond \exists x\ \varphi^*(x)$$

A nonstandard twist on the potential infinite

Rather than look at initial segments of $\mathbb{N}$ (which are badly behaved), look at nonstandard models of arithmetic, ordered by end-extension.

We can always produce new integers by end-extending to a bigger, better model of arithmetic.

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Before I answer this, let's backup for a general look at potentialist systems.

Potentialist systems

- There are other viewpoints which have this potentialist flavor, and we have a general framework for talking about them.
- For example, Zermelo can be interpreted as giving a potentialist view of sets: we can always climb upward to a larger inaccessible cardinal.

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Examples:

- (Zermelian potentialism) Worlds are V_κ for inaccessible κ, ordered by top-extension.
- (The generic multiverse) Worlds are forcing extensions of a fixed universe of sets, ordered by forcing extension.

The modal logic of a potentialist system

Given a potentialist system $\langle \mathcal{M}, \subseteq \rangle$, we can ask which modal assertions are valid for it.

Proposition

For any potentialist system, the modal theory S4 *is always valid.*

Axioms for S4:

$$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$$

$$\neg \Diamond p \Leftrightarrow \Box \neg p$$

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Proof.

Exercise :) □

Potentialist systems and their modal logics

- (Zermelian potentialism) Worlds are V_κ for inaccessible κ, ordered by top-extension.
  Has S4.3 as its modal validities. Add to S4 the axiom

$$(\Diamond p \wedge \Diamond q) \Rightarrow [(p \wedge \Diamond q) \vee (\Diamond p \wedge q)].$$

- (The generic multiverse) Worlds are forcing extensions of a fixed universe of sets, ordered by forcing extension.
  Has S4.2 as its modal validities. Add to S4 the axiom

$$\Diamond \Box p \Rightarrow \Box \Diamond p.$$

What does a potentialist system's modal logic tell us?

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For example the .2 axiom $\Diamond\Box p \Rightarrow \Box\Diamond p$ expresses a directedness to truth.

Think of the case with the generic multiverse:

- (Mostowski) The generic multiverse is not directed as a partial order.
- But its truth structure exhibits directedness: If forcing with $\mathbb{P}$ makes φ necessary, then φ is possible after forcing with $\mathbb{Q}$ by extending to an extension by $\mathbb{P} \times \mathbb{Q}$.

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In contrast, if a potentialist system has exactly S4 as its modal logic, that expresses an essential branchedness to truth. As you extend you make permantent choices you cannot take back.

Arithmetic potentialism

Woodin's universal algorithm can be used to exactly calculate the modal validities of arithmetic potentialism.

Theorem (Hamkins)

The modal validities for the potentialist system of models of arithmetic ordered by end-extension, allowing a single parameter in formulae, are precisely S4*. Indeed, the same is true if we order by arbitrary extension.*

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Proof idea.

If φ is not in S4 there is a finite pre-tree which invalidates φ. Use the universal algorithm to mimic the structure of this finite pre-tree within potential arithmetical truth. □

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You need the parameter (for the length of the sequence output by the universal algorithm). There are models of arithmetic whose modal validities for sentences are precisely S5 (add the axiom $\Diamond\Box p \Rightarrow p$ to S4).

From arithmetic to set theory

Question

Is there a universal algorithm for models of set theory? Can we use it to compute the modal validities of a corresponding potentialist system?

That is, is there an "algorithm" for models of set theory so that we can make the algorithm output whatever we like by extending to a larger model with new elements on the end?

Extensions of models of set theory

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- N is a end-extension of M if $y \in^N x \in M$ implies $y \in M$. That is, M is a transitive subclass of N.
- An end-extension N of M is covering if there is $m \in N$ so that $M \subseteq \{y \in N : y \in^N m\}$. Call m a cover of M.
- N is a top-extension of M if $x \in N \setminus M$ implies $\operatorname{rank} x \in N \setminus M$. That is, new elements have new ranks.

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The three notions are easily seen to be distinct.

- V end-extends L, but is not covering. Also, $\mathrm{V}[g]$ end-extends V but is not covering.
- Suppose κ is inaccessible and force to collapse V_κ to be countable. Then $\mathrm{V}[g]$ is a covering end-extension of V_κ but is not a top-extension.

Existence of end-extensions

Theorem (Keisler–Morley)

Every countable model of ZF *has an elementary end-extension.*

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Fact

There are uncountable models of ZF *without elementary end-extensions.*

By a theorem of Kaufmann, no Σ_3-elementary end-extension can be conservative. So no rather classless model of ZF has an elementary end-extension.

Absoluteness and extensions

Proposition

- *The assertions which are upward absolute for end-extensions are precisely the* Σ_1 *assertions.*
- *The assertions which are upward absolute for top-extensions are precisely the* Σ_2 *assertions.*

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- *The assertions which are upward absolute for end-extensions are precisely the Σ_1 assertions.*
- *The assertions which are upward absolute for top-extensions are precisely the Σ_2 assertions.*

Let me also recall some classical absoluteness results, formulated in the context of end-extensions of models of ZF.

- (Π^1_1 absoluteness) Let N be an end-extension of M. Then Π^1_1 statements about reals in M are absolute between M and N.
- (Shoenfield absoluteness) Let N be an end-extension of M with the same countable ordinals. Then Π^1_2 statements about reals in M are absolute between M and N.
- (Lévy absoluteness) Σ_1 statements about sets in $\mathrm{L}_{\omega_1^{\mathrm{L}}}$ are absolute between $\mathrm{L}_{\omega_1^{\mathrm{L}}}$ and V.

The Σ_2 universal finite sequence for top-extensions

Theorem (Hamkins–Woodin)

There is a Σ_2 definition of a finite sequence with the following properties.

1. ZFC *proves the sequence is finite.*
2. *In any transitive model of* ZFC *the sequence is finite.*
3. *If M is a countable model of* ZFC *in which the sequence is s and $t \in M$ is a finite set extending s, then there is a top-extension N of M in which the sequence is t.*

Observe you can just as well ask for a universal set, say by taking the union of the elements on the sequence.

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Corollary (Hamkins–Woodin)

The modal validities of the potentialist system consisting of countable models of set theory ordered by top-extension are precisely S4.

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Is there a Σ_1 definition for a finite sequence which is universal for end-extensions?

As the title of this talk suggests, the answer is yes. I want to spend the remaining time telling you about the Σ_1 universal finite sequence and what you can do with it.

The Σ_1 universal finite sequence for end-extensions

Let $\overline{\mathrm{ZF}}$ be a fixed computably enumerable extension of ZF.

Theorem (Hamkins–W.)

There is a Σ_1 definition for a finite sequence

$$a_0, \ldots, a_n$$

with the following properties.

1. ZF *proves the sequence is finite.*
2. *If $M \models \overline{\mathrm{ZF}}$ is transitive then the sequence in M is the empty sequence.*
3. *If in countable $M \models \overline{\mathrm{ZF}}$ the sequence is s and $t \in M$ is any finite extension of s, then there is $N \models \overline{\mathrm{ZF}}$ an end-extension of M so that the sequence in N is exactly t.*

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4. *Indeed, in* (3) *it suffices that $M \models \mathrm{ZF}$ has an inner model $W \models \overline{\mathrm{ZF}}$.*

Overview of the argument

- It is convenient to think of terms of processes which produce a sequence step by step.
- Must then check that the processes are Σ_1-definable and that they have the extension property.
- The ω-nonstandard and ω-standard cases will be handled separately with different processes, and at the end we will check how they can be combined into a single process that works for all models.

Process A—for ω-nonstandard models

$a_0, \ldots, a_n$ is defined using auxiliary information $k_0 > \cdots > k_n$ finite ordinals and $m_0 \in \cdots \in m_n$ countable transitive sets.

Stage n succeeds if all previous stages succeed, and

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If stage n succeeds, let (a_n, k_n, m_n) be the triple seen first in the L-order.

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A triple (a, k, m) witnessing success is a Δ_1 property, since it is Π^1_1 to say there is no such N and so we can check it by looking at the ordinal rank of the canonically associated well-founded tree.

By Lévy absoluteness, if such triple exists then there is one in $\mathrm{L}_{\omega_1^{\mathrm{L}}}$.

So the map $n \mapsto (a_n, k_n, m_n)$ is Σ_1.

Some observations about Process A

Definition

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- The sequence is finite, because the k_i count down.
- Each k_i must be nonstandard, by an easy reflection argument in W.
- In particular, if M is ω-standard then no stage succeeds, so the sequence is empty.

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- Each k_i must be nonstandard, by an easy reflection argument in W.
- In particular, if M is ω-standard then no stage succeeds, so the sequence is empty.

It remains to check the extension property, that we can end-extend any countable model M to put whatever finitely many elements of M on the end of the sequence. Observe that it suffices to check the $+1$ extension property where we add only one new element.

The $+1$ extension property for Process A

Definition

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Suppose stage n is the first failure in M. Consider M^+, a countable elementary end-extension of M. Let $M^+[g]$ be a forcing extension which collapses $m = {\mathrm{V}_\theta}^{M^+}$ to be countable, where θ is an ordinal above M. Because M^+ satisfies the Π^1_2 statement that stage n fails, so does $M^+[g]$.

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Thus inside $M^+[g]$: for any $a \in M$ and nonstandard $k < k_{n-1}$ one can find a model N of $\overline{\mathrm{ZF}}_k$, which end-extends m and whose process A sequence is exactly $a_0, \ldots, a_{n-1}, a$, using the same auxiliary information.

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Thus inside $M^+[g]$: for any $a \in M$ and nonstandard $k < k_{n-1}$ one can find a model N of $\overline{\mathrm{ZF}}_k$, which end-extends m and whose process A sequence is exactly $a_0, \ldots, a_{n-1}, a$, using the same auxiliary information.

So $N \models \overline{\mathrm{ZF}}$ is the desired end-extension of M.

Process B—for ω-standard models

$a_0, \ldots, a_n$ is defined using auxiliary information $\lambda_0 > \cdots > \lambda_n$ countable ordinals and $m_0 \in \cdots \in m_n$ countable transitive sets.

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The tree canonically associated to the Π^1_1 assertion "$(m, \in)$ has **no** end-extension blah blah" is well-founded and has rank λ.

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Stage n succeeds if all previous stages succeed, and there are a, $\lambda < \lambda_{n-1}$, and $m \ni m_{n-1}$ so that $(m, \in)$ has **no** end-extension to a model N of $\overline{\text{ZF}}$ in which the process B sequence is exactly $a_0, \ldots, a_{n-1}, a$, defined using the same auxiliary information; and

The tree canonically associated to the Π^1_1 assertion "$(m, \in)$ has **no** end-extension blah blah" is well-founded and has rank λ.

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Similar to before: The apparent circularity of the definition is resolved by the Gödel–Carnap fixed-point lemma; The definition is Σ_1; The sequence is finite, because the λ_i count down; Each λ_i must be nonstandard.

This last claim needs justification.

Each λ_i is nonstandard

Definition

Stage n succeeds if all previous stages succeed, and there are a, $\lambda < \lambda_{n-1}$, and $m \ni m_{n-1}$ so that $(m, \in)$ has **no** end-extension to $N \models \overline{\text{ZF}}$ where the process B sequence is exactly $a_0, \ldots, a_{n-1}, a$, using the same auxiliary information, and the tree for this Π^1_1 assertion is well-founded with rank λ. If stage n succeeds, let (a_n, λ_n, m_n) be the triple seen first in the L-order.

Suppose M is a countable model of ZF with an inner model $W \models \overline{\text{ZF}}$ in which n is the last successful stage. By Lévy absoluteness W agrees with M on what Process B does. In particular, W thinks that $(m_n, \in)$ has **no** end-extension to a model of $\overline{\text{ZF}}$ in which the sequence is exactly $a_0, \ldots, a_{n-1}, a_n$, etc.

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But externally, we can see that W is itself such an end-extension. So the "well-founded" tree in W witnessing the truth in W of this assertion cannot actually be well-founded, and so its rank λ_n must be nonstandard.

The $+1$ extension property for Process B

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As before: Let n be the first failure in M and consider $M^+[g]$ a forcing extension of M^+, a countable elementary end-extension of M, in which some large enough $m = \mathrm{V}_\theta{}^{M^+}$ is collapsed to be countable. Fix $a \in M$.

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Suppose toward a contradiction that $(m, \in^{M^+[g]})$ has **no** end-extension to $N \models \overline{\mathrm{ZF}}$ in which a is placed on the sequence at stage n as the last successful stage. Then $M^+[g]$ must agree there is no such N. This is a Π^1_1 assertion about the data, so in $M^+[g]$ there is some λ the rank of the corresponding well-founded tree.

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We assumed that this Π^1_1 assertion is true externally, so it follows that λ must be in the well-founded part of $M^+[g]$. In particular, $\lambda < \lambda_{n-1}$, since λ_{n-1} was ill-founded.

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Therefore, $M^+[g]$ thinks there are suitable m and a so that the assertion "$(m, \in)$ has **no** end-extension to a model of $\overline{\mathrm{ZF}}$ in which a is placed on the sequence at stage n, the last successful stage" has rank λ, and $\lambda < \lambda_{n-1}$. This is a Σ_1 assertion about λ and the other data.

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$M^+[g]$ thinks there are suitable m and a so that the assertion "$(m, \in)$ has **no** end-extension to a model of $\overline{\mathrm{ZF}}$ in which a is placed on the sequence at stage n, the last successful stage" has rank $\lambda < \lambda_{n-1}$, a Σ_1 assertion.

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(Case 1: λ is countable in L^{M^+}.) All parameters in the Σ_1 assertion are countable in L^M. So by Lévy reflection in $M^+[g]$ we get the statement is true in L^{M^+} and hence also in L^M. But then stage n was successful in M, a contradiction.

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(Case 2: λ is uncountable in L^{M^+}.) L^M is well-founded beyond its ω_1, as λ is well-founded. So no stage succeeds in M, as the $\lambda_i < \omega_1^{\mathrm{L}^M}$ must be nonstandard. The Σ^1_2 assertion "there is suitable m with **no** end-extension to a model where the sequence is a" is true in $M^+[g]$, hence true in L^{M^+}, hence true in L^M. So M has a successful stage, a contradiction.

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Either way we get a contradiction. So $m = \mathrm{V}_\theta{}^{M^+}$ really does have the desired end-extension, which must also be an end-extension of M, showing the $+1$ extension process.

Process C—for all models

Again proceed in stages. Put a new element a_n on the sequence for either an A-reason or a B-reason. A-reasons are given by data (a_n, k_n, m_n), and B-reasons are given by data (a_n, λ_n, m_n), and we look for the first witness in the L-order. Once an A-reason is successful—hence we know we are in an ω-nonstandard model—only look for A-reasons from hereon out.

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Can now check that this combined Process C has the $+1$ extension property.

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Can now check that this combined Process C has the $+1$ extension property. □

Must the models be countable?

For the main theorem, we require the models of $\overline{\mathsf{ZF}}$ to be countable, in order to use the Keisler–Morley theorem and know that forcing extensions always exist. Can we remove that assumption?

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No.

Observation

If $M \models \mathsf{ZF}$ contains all countable ordinals then by Lévy absoluteness no end-extension of M can have new Σ_1 facts.

L-extensions

If $M \subseteq N \models \mathrm{V} = \mathrm{L}$, call N an L-extension of M if M is an initial segment in the L_α-hierarchy for N.

Compare: N is a top-extension of M if M is an initial segment in the V_α-hierarchy for N.

Corollary (Hamkins–W.)

There is a Σ_1 definition for a finite sequence with the following properties.

1. $\mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ *proves the sequence is finite.*
2. *In any transitive model the sequence is empty.*
3. *If $M \models \mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ is a countable model in which the sequence is s and $t \in M$ is a finite sequence extending s then there is $N \models \mathsf{ZFC} + \mathrm{V} = \mathrm{L}$ an L-extension of M in which the sequence is exactly t.*

The modal validities of end-extensional potentialism

Corollary (Hamkins–W.)

Consider the potentialist system consisting of countable models of $\overline{\mathsf{ZF}}$ *ordered by end-extension. The modal validities of this potentialist system are precisely* S4.

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The proof follows the corresponding argument for arithmetic potentialism.

A more refined theorem

Suppose S and T have the following properties.

- S and T extend KP.
- T is computably enumerable.
- Each finite subtheory of T reflects cofinally in the transitive sets in any model of T.
- Countable models of S satisfy Shoenfield absoluteness.
- Every countable model of S has a Σ_1-elementary covering end-extension, which also satisfies Shoenfield absoluteness.

Then there is a Σ_1 definable universal finite sequence for countable models of S with inner models of T, end-extending to models of T. (That is, T plays the role of $\overline{\text{ZF}}$ and S plays the role of ZF.)

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For example, $S = T = \text{ZF}^-$ qualifies.

Resurrection in end-extensions

Ignoring the part about the universal finite sequence, the main theorem says that we can resurrect properties of inner models in end-extensions: Any countable model of S with an inner model of T can be end-extended to a model of T.

Corollary

- *(Barwise extension theorem) Any countable model of* ZF *end-extends to a model of* $\mathsf{ZFC} + \mathrm{V} = \mathrm{L}$.
- *Any countable model of* ZFC *with a measurable cardinal end-extends to a model of* $\mathsf{ZFC} + \mathrm{V} = \mathrm{L}[\mu]$.
- *Any countable model of* ZFC *which has infinitely many Woodin cardinals with a measurable above end-extends to a model of* $\mathsf{ZF} + \mathsf{AD} + \mathrm{V} = \mathrm{L}(\mathbb{R})$.

Characterizing end-extensional possibility

Theorem (Hamkins–W.)

Consider a countable ω-nonstandard model of $\overline{\mathsf{ZF}}$. For any assertion $\varphi(a)$ about a hereditarily countable object $a \in M$ the following are equivalent.

1. *$M \models \Diamond\varphi(a)$ in the end-extensional potentialist system.*
2. *$M \models \Diamond\varphi(a)$ in the Δ_0-elementary potentialist system.*
3. *For each countable transitive $m \in M$ with $a \in^M m$ and each standard k, M thinks that $\langle m, \in^M \rangle$ end-extends to a model $N \models \overline{\mathsf{ZF}}_k + \varphi(a)$.*
4. *For each real $x \in M$ and each standard k, M thinks that there is an ω-standard model of $\overline{\mathsf{ZF}}_k + \varphi(a)$ which contains x.*
5. *(For sentences) φ is consistent with $\overline{\mathsf{ZF}}$ plus the Σ_1-theory of M.*

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5. *(For sentences) φ is consistent with $\overline{\mathsf{ZF}}$ plus the Σ_1-theory of M.*

(Can also formulate a more refined version of this.)

Maximal Σ_1-theories

A set T of Σ_1-sentences is a maximal Σ_1-theory over $\overline{\mathsf{ZF}}$ if $\overline{\mathsf{ZF}} + T$ is consistent and if φ is any Σ_1-sentence not in $\overline{\mathsf{ZF}} + T$ then $\overline{\mathsf{ZF}} + T + \varphi$ is not consistent.

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Observation

- *Every Σ_1-theory consistent with $\overline{\mathsf{ZF}}$ extends to a maximal Σ_1-theory.*

Maximal Σ_1-theories

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Observation

- *Every Σ_1-theory consistent with $\overline{\mathsf{ZF}}$ extends to a maximal Σ_1-theory.*
- *If $M \models T$ where T is a maximal Σ_1-theory then M must be ω-nonstandard, and indeed its universal finite sequence has nonstandard length. (Because the assertions "the universal finite sequence has length $\geq k$" are Σ_1-sentences for standard k.)*

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Corollary (Hamkins–W.)

Every countable model of $\overline{\mathsf{ZF}}$ has a Δ_0-elementary extension to a model satisfying the end-extensional maximality principle.

In general we cannot ask the extension to be an end-extension, for example with any ω-standard model.

It's always nice to end with an open question :)

Question

Characterize which countable ω-nonstandard models of $\overline{\mathsf{ZF}}$ end-extend to a model of the end-extensional maximality principle.

Equivalently, characterize which countable ω-nonstandard models of $\overline{\mathsf{ZF}}$ end-extend to a model of a maximal Σ_1-theory.

Thank you!

Some references

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