1. (5 points) State the version of Fubini's theorem for integrating a function $f(x, y)$ over a rectangular region $R$ given by $a \le x \le b$ and $c \le y \le d$. (Hint: remember that you need to make an assumption about $f(x, y)$.)

Suppose $F(x,y)$ is continuous on R. Then,

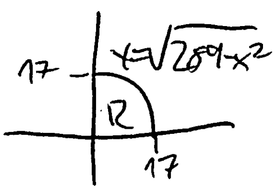
$$\iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx$$

$$= \int_c^d \int_a^b F(x,y)\,dx\,dy$$

2. (15 points) Set up **but do not solve** two different iterated integrals equal to the double integral

$$\iint_R \frac{e^{xy}}{x}\, dA,$$

where $R$ is the quarter circle of radius 17 in the first quadrant. One of these integrals must be in rectangular form, the other must be in polar form. (Hint: $17^2 = 289$.)




$$\int_0^{17}\int_0^{\sqrt{289-x^2}} \frac{e^{xy}}{x}\,dy\,dx \quad \text{or} \quad \int_0^{17}\int_0^{\sqrt{289-y^2}} \frac{e^{xy}}{x}\,dx\,dy$$

$0 \le \theta \le \pi/2$

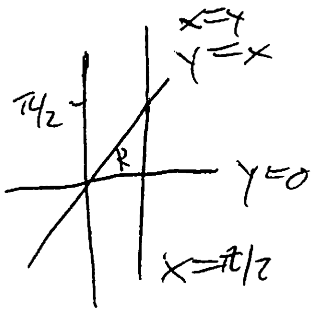
$0 \le r \le 17$

$$\int_0^{\pi/2}\int_0^{17} \frac{e^{r^2\cos\theta\sin\theta}}{r\cos\theta}\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{17} \frac{e^{r^2\cos\theta\sin\theta}}{\cos\theta}\,dr\,d\theta$$

or

$$\int_0^{17}\int_0^{\pi/2} \frac{e^{(r^2\cos\theta\sin\theta)}}{\cos\theta}\,d\theta\,dr$$

3. (20 points) Consider the double integral $\iint_R \sin(x-y)\,dA$ where $R$ is the triangle bounded by the lines $y = x$, $x = \pi/2$, and $y = 0$. Set up two different iterated integrals to calculate this double integral, one where you integrate with respect to $x$ and then with respect to $y$, and the other where you integrate with respect to $y$ and then with respect to $x$. Solve one of these two—your choice!—integrals.




$$(*)\quad \int_0^{\pi/2}\int_0^{x} \sin(x-y)\,dy\,dx$$

$$(\dagger)\quad \int_0^{\pi/2}\int_y^{\pi/2} \sin(x-y)\,dx\,dy$$

$$(*) = \int_0^{\pi/2} +\cos(x-y)\Big|_{y=0}^{y=x} dx = \int_0^{\pi/2} -\cos x + \underbrace{\cos 0}_{=1}\,dx = -\sin x + x\Big|_{x=0}^{x=\pi/2}$$

$$= \underline{-1+\pi/2}$$

$$(\dagger) = \int_0^{\pi/2} -\cos(x-y)\Big|_{x=y}^{x=\pi/2} dy = \int_0^{\pi/2} -\cos(-y+\pi/2) + \underbrace{\cos 0}_{=1}\,dy = -\sin(\pi/2-y)+y\Big|_0^{\pi/2}$$

$$= \underline{\pi/2 - 1}$$

4. (10 points) Calculate the double integral

$$\iint_R xy + 4\,\mathrm{d}A$$

over the rectangle $R$ given by $-2 \le x \le 2$ and $-2 \le y \le 2$.

$$\int_{-2}^{2}\int_{-2}^{2} xy+4\,dy\,dx = \int_{-2}^{2} \frac{xy^2}{2}+4y\Big|_{y=-2}^{y=2} dx = \int_{-2}^{2} 2x+\cancel{8}-2x+\cancel{8}\,dx$$

$$= \int_{-2}^{2} 16\,dx = 16\cdot 4 = \underline{64}$$

5. (10 points) Consider the point $P = (0, 2, -1)$ and the line given by $\vec{\ell}(t) = (5\vec{\imath} - \vec{\jmath} + 2\vec{k})t$. Find an equation for the plane determined by the point $P$ and the line $\vec{\ell}(t)$.

NB: Several — many — students misread problem and interpreted $\vec{l}(t)$ as normal to the plane. Say something when you hand this back.

$t=0 \Rightarrow \vec{l}(0)=(0,0,0)=:Q$

$\vec{PQ}=(0,2,-1)$ $\vec{d}=(5,-1,2)$

$$\vec{PQ}\times\vec{d}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 2 & -1\\ 5 & -1 & 2\end{vmatrix} = \begin{matrix}4\vec{i}-5\vec{j}+0\vec{k}\\ -\vec{i}-0\vec{j}-10\vec{k}\end{matrix} = 3\vec{i}-5\vec{j}-10\vec{k}=(3,-5,-10)$$

So Equation is $3(x-0)-5(y-0)-10(z-0)=0$

$\underline{3x-5y-10z=0}$

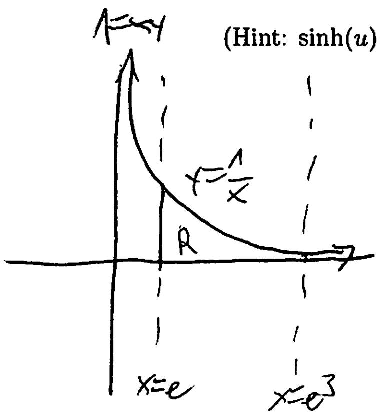
OR: $3(x-0)-5(y-2)-10(z+1)=0$

$\underline{3x-5y-10z=0}$

6. (20 points) Let $R$ be the region bounded by the curve $xy = 1$ and the lines $y = 0$, $x = e$, and $x = e^3$. Sketch the region $R$ and calculate the double integral

$$\iint_R \sinh(xy)\, \mathrm{d}A.$$

(Hint: $\sinh(u) = \frac{e^u - e^{-u}}{2}$ is the *hyperbolic sine.*)




$$e \le x \le e^3 \qquad 0 \le y \le \frac{1}{x}$$

$$\int_e^{e^3}\int_0^{1/x} \frac{e^{xy} - e^{-xy}}{2}\, dy\, dx = (*)$$

$$(*) = \int_e^{e^3} \frac{1}{2}\left(\frac{e^{xy}}{x} + \frac{e^{-xy}}{+x}\right)\Bigg|_{y=0}^{y=\frac{1}{x}} dx = \int_e^{e^3} \frac{1}{2}\left(\frac{e}{x} + \frac{e^{-1}}{x} - \frac{2}{x}\right) dx$$

$$= \int_e^{e^3} \frac{e+e^{-1}-2}{2} \cdot \frac{1}{x}\, dx = \frac{e+e^{-1}-2}{2} \cdot \overbrace{\left(\log e^3 - \log e\right)}^{3-1=2} = \boxed{e+e^{-1}-2}$$

<u>NB:</u> $\int \sinh u\, du = -\cosh u$, where $\cosh(u) = \frac{e^u + e^{-u}}{2}$ is the hyperbolic cosine.

1. (6 points) Consider the hyperbolic paraboloid given by the equation

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

where $a$ and $b$ are positive real numbers. What is the average height of this hyperbolic paraboloid over the rectangular region $R$ given by $0 \le x \le a$ and $0 \le y \le b$?

Area = $\underline{ab}$, by geometry

$$\int_0^b \int_0^a \frac{x^2}{a^2} - \frac{y^2}{b^2} \, dx\,dy = \int_0^b \frac{x^3}{3a^2} - \frac{y^2}{b^2}x \Big|_{x=0}^{x=a} dy = \int_0^b \frac{a}{3} - \frac{y^2}{b^2} \cdot a \, dy$$

$$= \frac{a}{3}y - \frac{ay^3}{3b^2}\Big|_{y=0}^{y=b} = \frac{ab}{3} - \frac{ab}{3} - 0 + 0 = \underline{0}$$

So Av val $= \dfrac{0}{ab} = \underline{0}$

2. (6 points) Let $m < M$ be real numbers. What is the average height of the same hyperbolic paraboloid from the previous question over the rectangular region $R$ given by $ma \le x \le Ma$ and $mb \le y \le Mb$? (So the previous question was the special case where $m = 0$ and $M = 1$. You can check your work by checking that the answer you get here matches the previous answer for this special case.)

$$A_{rea} = (Ma - ma)(Mb - mb) = ab(M-m)^2$$

$$\int_{mb}^{Mb}\int_{ma}^{Ma} \frac{x^2}{a^2} - \frac{y^2}{b^2}\,dx\,dy = \int_{mb}^{Mb} \frac{x^3}{3a^2} - \frac{y^2x}{b^2}\Big|_{x=ma}^{x=Ma} dy = \int_{mb}^{Mb} \frac{M^3a}{3} - \frac{May^2}{b^2} - \frac{m^3a}{3} + \frac{may^2}{b^2}\,dy$$

$$= \int_{mb}^{Mb} \frac{a}{3}(M^3-m^3) \cancel{A} - a(M-m)\frac{y^2}{b^2}\,dy = \left[\frac{a}{3}(M^3-m^3)y - \frac{a(M-m)y^3}{3b^2}\right|_{y=mb}^{y=Mb}$$

$$= \frac{a}{3}(M^3-m^3)(Mb-mb) - \frac{a(M-m)M^3b}{3} + \frac{a(M-m)m^3b}{3}$$

$$= \frac{ab}{3}(M^3-m^3)(M-m) - \frac{ab}{3}(M-m)(M^3-m^3)$$

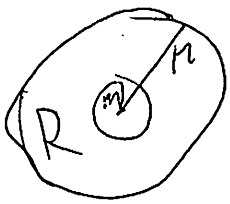
$$= 0$$

So average value is $\frac{0}{ab(M-m)^2} = \underline{0}$

3. (8 points) Let $0 \leq m < M$ be non-negative real numbers. Consider the hyperbolic paraboloid given by the equation

$$z = y^2 - x^2.$$

What is the average height of this hyperbolic paraboloid on the annulus region $R$ consisting of all points whose distance to the origin is between $m$ and $M$? (Hint: recall that $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.)




$m \leq r \leq M$

$0 \leq \theta \leq 2\pi$

$$y^2 - x^2 = r^2 \sin^2\theta - r^2\cos^2\theta$$

$$= r^2(\sin^2\theta - \cos^2\theta)$$

$$= -r^2\cos(2\theta)$$

$$\int_0^{2\pi}\int_m^M -r^3\cos(2\theta)\,dr\,d\theta = \int_0^{2\pi} \cos(2\theta) \cdot -\frac{r^4}{4}\Big|_m^M d\theta = \int_0^{2\pi} \frac{m^4 - M^4}{4} \cdot \cos(2\theta)\,d\theta$$

$$= \frac{m^4 - M^4}{4} \cdot \frac{\sin(2\theta)}{2}\Big|_0^{2\pi} = 0 - 0 = \underline{0}$$

So Av val $= \underline{0}$