# Math 321: More with functions: equinumerosity

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# Bijections

Recall that a function $f : A \to B$ is a bijection onto $B$ if $f$ is both one-to-one and onto $B$. That is, $f$ satisfies the following property:

- For all $b \in B$ there is a unique $a \in A$ so that $f(a) = b$.

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Today we're going to discuss a use of bijections to talk about sets.

# Comparing discrete collections

Suppose you have two discrete collections $A$ and $B$ of objects. How do you determine whether your two collections have the same size?

# Equinumerosity

## Definition

Two sets $A$ and $B$ are equinumerous, let's denote this as $A \sim B$, if there is a bijection $f : A \to B$.

This definition is meant to capture the idea that $A$ and $B$ have the same number of elements.

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($\sim$ is symmetric) If $f : A \to B$ is a bijection from $A$ onto $B$ then $f^{-1} : B \to A$ is a bijection from $B$ onto $A$.

($\sim$ is transitive)

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**Proposition**

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($\sim$ is reflexive) For any set $A$, the identity function $\mathrm{id}_A : A \to A$ is a bijection from $A$ onto $A$.
($\sim$ is symmetric) If $f : A \to B$ is a bijection from $A$ onto $B$ then $f^{-1} : B \to A$ is a bijection from $B$ onto $A$.
($\sim$ is transitive) Suppose $f : A \to B$ and $g : B \to C$ are bijections. Then $g \circ f : A \to C$ is a bijection. □

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The answer is that these ideas let us get a handle on infinite sets. With finite sets, things are more straightforward, but it's not so clear what to do with infinite sets like $\mathbb{N}$ or $\mathbb{Q}$ or $\mathbb{R}$. This extra abstraction lets us talk about a more general context.

# Finite and infinite sets

## Definition

A set $A$ is finite if there is some $n \in \mathbb{N}$ so that

$$A \sim \{k \in \mathbb{N} : k < n\} = \{0, 1, \ldots, n-1\}.$$

If $A$ is not finite then we call it infinite.

# Cardinality

Informally, the cardinality of a set is the number of elements in the set. Formally, we define this using equivalence classes.

## Definition

Given a set $A$, the cardinality of $A$, denoted $|A|$, is the equivalence class $[A]_\sim$ with respect to the equinumerosity relation.
So given sets $A$ and $B$, we have $|A| = |B|$ iff $A \sim B$.

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So given sets $A$ and $B$, we have $|A| = |B|$ iff $A \sim B$.

For finite sets, we use the familiar names for natural numbers for their cardinalities. That is, we will simply write $|A| = n$ to mean $A \sim \{k \in \mathbb{N} : k < n\}$.

# Some infinite sets with cardinality $|\mathbb{N}|$

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To be continued in the next lecture...

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- This means $A \sim B$, which means there is a bijection $f : A \to B$.

Can we define what it means to say $|A| \leq |B|$?

- Say $|A| \leq |B|$ if there is a one-to-one function $f : A \to B$.
- Say $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

# The pigeonhole principle

**Fact (The pigeonhole principle)**

*If $|A| < |B|$ then no function $f : B \to A$ can be one-to-one.*

If you have more pigeons than holes, then at least one pigeonhole must contain multiple pigeons.