Math 321: Summing up a semester

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Let's sum up some of what this semester was about.

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4. That might be enough for you to see what to do, but in general it probably won't. So ask: what facts about these sorts of objects do I already know? (This is a good place to reference your notes or textbook.)

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Thinking about this a bit, you might stumble on the $n = 2$ case as the key one to think about:

If x and y are two points in a linear order, then by the trichotomy property of linear orders, either $x \leq y$ or $y \leq x$. So the smaller is the minimum and the larger is the maximum.

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For the inductive step, assume that any linear order with n points has a maximum and a minimum. Consider a linear order with the points $x_1, \dots, x_n, x_{n+1}$. By the inductive hypothesis, looking at just the points $x_1, \dots, x_n$ they have a maximum M_0 and a minimum m_0. So $M = \max(M_0, x_{n+1})$ is the maximum and $m = \min(m, x_{n+1})$ is the minimum of the $n + 1$ many points.

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Conjunctions: Many math statements are combinations joined by "and"s. (For example, f is a bijection means f is an injection *and* f is a surjection.)

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Sometimes, you may be able to directly show P or else directly show Q.

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Some examples from the semester:

- To prove $\sqrt{2}$ is irrational we assumed it was rational—that is, we assumed $\sqrt{2} = p/q$ for integers p, q—and we derived a contradiction—p and q both had no common factors but also had 2 as a common factor.
- To prove $\mathbb{R}$ is uncountable we assumed it was countable—that is, we assumed there was an enumeration $x_0, x_1, \ldots$ of all reals—and we derived a contradiction—we found a real d not on the enumeration.

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- At the outermost level, this is an iff statement, so we need to independently prove two things:
    - If A is countable then $A = \emptyset$ or there is a surjection $f : \mathbb{N} \to A$.
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  - Assume we have a countable set A—that is, assume there is an injection $g : A \to \mathbb{N}$—and try to prove if $A \neq \emptyset$ then there is a surjection $f : \mathbb{N} \to A$.
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  - If $A = \emptyset$ or there is a surjection $f : \mathbb{N} \to A$, then A is countable.
  - Prove this by cases: If $A = \emptyset$ then A is countable; and if there is a surjection $f : \mathbb{N} \to A$ then A is countable.

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- But it's also important to understand and use the precise formal definition.
- If you don't remember the exact statement of a definition, look in your notes or textbook!