MATH 321: HOMEWORK 6
DUE FRIDAY, MAR 5 BY 11:00PM

This homework is about *Khayyam's triangle*, named after the eleventh century Persian mathematician Omar Khayyam.[1] You may have seen it before, but let me give the definition in case you have not. This triangle of numbers is connected to the *binomial coefficients* $\binom{n}{k}$, as you will prove in this homework. It is constructed row by row, in a recursive procedure starting at row 0, which contains just one number, namely 1. Thereon, row n has $n+1$ many numbers: the two on the outside are 1s, and the interior numbers are the sum of the two numbers above:

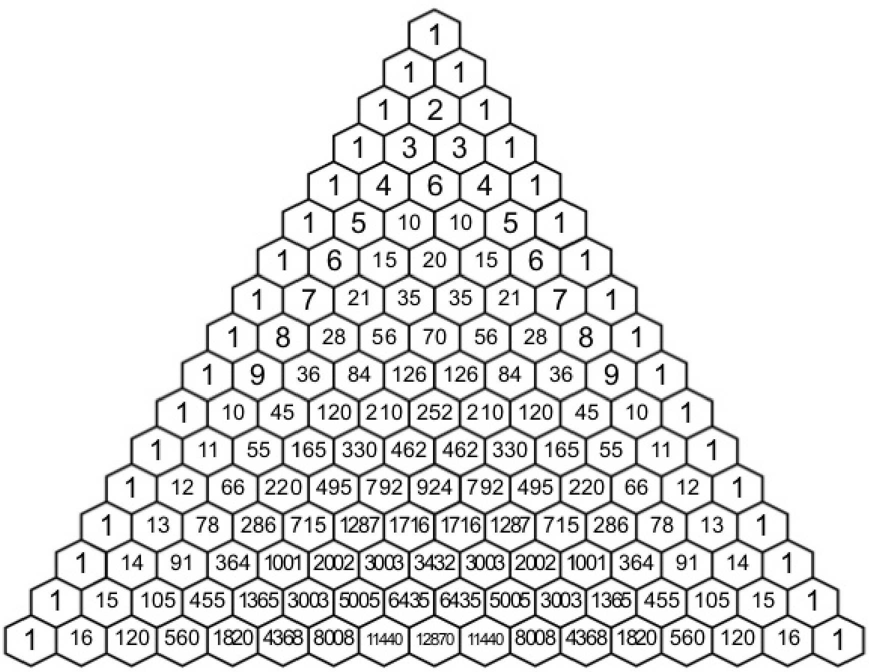

FIGURE 1. The first several rows of Khayyam's triangle.

Let me also remind you that $\binom{n}{k}$ is, by definition, the number of way to pick k objects from n choices, where you don't care about the order (we assume $0 \leq k \leq n$). The formula to directly compute $\binom{n}{k}$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(If you are using LaTeX, the command for binomial coefficients is `\binom{n}{k}`. Make sure you have `\usepackage{amsmath}` in the preamble to your file, as that is the package which defines that command.)

[1] The triangle was earlier studied by the mathematician Al-Karaji, and it was independently discovered elsewhere, e.g. in China and Europe. Europeans usually call it *Pascal's triangle*, after Blaise Pascal.

Problem 1. Prove by induction on n that the kth number in the nth row of Khayyam's triangle is $\binom{n}{k}$, where each row is indexed starting at 0.[2] [Hint: You need to show two things, since the rules for exterior versus interior positions are different: (a) that the exterior positions are $\binom{n}{0} = \binom{n}{n} = 1$, which include the base case $\binom{0}{0} = 1$, and for the inductive step to get the interior positions, (b) that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.]

Problem 2. Use the result of the previous problem to prove by induction that the following sum holds for all natural numbers n:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \cdots + \binom{n}{n} = 2^n.$$

Problem 3. Prove by induction on $n \geq 1$ that $\binom{n+1}{2} = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n$.

Problem 4. Prove that if p is prime, then every number in the pth row of Khayyam's triangle is either 1 or a multiple of p.

[2] For example, in row 3 the 0th number is 1, the 1st number is 3, the 2nd number is 3, and the 3rd number is 1.