Math 321: Infinity, II

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Spring 2021

Last week

- A set X is countable if there is a one-to-one function $f : X \to \mathbb{N}$.
- Equivalently, X is countable if you can enumerate all the elements of X.
- We saw lots of different sets are countable: $\mathbb{N}^k$, $\mathbb{N}^*$, $\mathbb{Z}$, $\mathbb{Q}$.
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- Given any enumeration of real numbers we can find diagonalize against the enumeration to produce a real number not on the enumeration.

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Let's see some more uncountable sets.

More uncountable sets

For a set A, its powerset $\mathcal{P}(A)$ is the set of subsets of A.

Theorem (Cantor, 1891)

For any set A there is no one-to-one function $f : \mathcal{P}(A) \to A$.

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Let's do this by contradiction. Suppose that $f : \mathcal{P}(A) \to A$ is one-to-one. Define a set $D \subseteq A$ as

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Corollary

$\mathcal{P}(\mathbb{N})$ is uncountable

A bit of the general theory

Let A and B be sets.

- A and B are equinumerous, written $A \simeq B$, if there is a bijection from A to B.
- $A \lesssim B$ if there is a one-to-one function from A to B.

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- If $A \simeq \mathbb{N}$ we call A countably infinite.

Because every countable set is equinumerous with a subset of $\mathbb{N}$, it is enough to consider $A \subseteq \mathbb{N}$. If A is finite, then it's finite. If A is infinite, we get a bijection $f : \mathbb{N} \to A$ by setting $f(n)$ to be the n-th element of A, according to the order on $\mathbb{N}$. This function is defined on all of $\mathbb{N}$ because A is infinite.

Our old results in this new language

We can use these new definitions to succinctly state some of the earlier results.

- $\mathbb{N} \simeq \mathbb{N} \times \mathbb{N} \simeq \mathbb{N}^*$
- $\mathbb{N} \simeq \mathbb{Z} \simeq \mathbb{Q}$
- $\mathbb{N} < \mathbb{R}$ (i.e. $\mathbb{N} \lesssim \mathbb{R}$ but $\mathbb{N} \not\simeq \mathbb{R}$)
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Iterating out this last one we get infinitely many different sizes of infinite sets:

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And we could build out even higher. If

$$\mathcal{P}^\omega(\mathbb{N}) = \bigcup_{k \in \mathbb{N}} \mathcal{P}^k(\mathbb{N})$$

then $\mathcal{P}^\omega(\mathbb{N}) > \mathcal{P}^n(\mathbb{N})$ for every $n \in \mathbb{N}$.

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And can then keep going:

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The Cantor–Schröder–Bernstein theorem

Theorem (Cantor–Schröder–Bernstein)

If $A \lesssim B \lesssim A$ *then* $A \simeq B$.

Theorem (Cantor–Schröder–Bernstein)

If $A \precsim B \precsim A$ *then* $A \simeq B$.

It may be tempting to think this result is obvious—if A is at least as big as B and B is at least as big as A then surely they must have the same size. But it's not obvious. The proof is nontrivial.

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It's easy to see $\mathbb{R} \lesssim \mathbb{R}^2$ (why?), so let's check the other direction. We need a one-to-one function $f : \mathbb{R}^2 \to \mathbb{R}$.

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It's usually easier to construct two one-to-one functions rather than get an exact bijection, so this theorem is nice :)

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If there are one-to-one functions $f : A \to B$ *and* $g : B \to A$ *then there is a bijection* $h : A \to B$.

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So h is a bijection, as desired. □

Some equinumerosities

Theorem

The following sets are all equinumerous.

1. $\mathbb{R}$;
2. *Any nondegenerate interval of real numbers;*
3. $\mathcal{P}(\mathbb{N})$;
4. $\mathbb{N}^{\mathbb{N}}$, *the set of functions* $\mathbb{N} \to \mathbb{N}$;
5. $\mathbb{R}^{\mathbb{N}}$, *the set of functions* $\mathbb{N} \to \mathbb{R}$;
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It suffices to construct one-to-one functions in both directions, not to directly construct bijections.