# Math 321: Some logic

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# Previously in Math 321

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So let's talk a bit about logic.

# Basic objects of logic

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- In logic, the objects we care about are linguistic.
- A predicate is a statement that can be either true or false. We will care about mathematical statements, but logic applies more broadly.
    - "There are infinitely many primes."
    - "There exists a real-valued function which is continuous everywhere but differentiable nowhere."
    - "$p$ is prime."
    - "The limit of $f(x)$ as $x$ approaches $a$ is $L$."

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  - "The limit of $f(x)$ as $x$ approaches $a$ is $L$."
- Some predicates have parameters, others do not. When we use variables to refer to predicates, we write e.g. $P(x, y)$ to denote that the predicate $P$ has parameters $x$ and $y$.
- Whether $P(x, y)$ is true can depend on what you plug in for $x$ and $y$!

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So let's analyze how this works. We'll start with predicates without parameters.

# Truth tables

One way to analyze this is to use truth tables, developed circa 1920 by the philosopher Ludwig Wittgenstein and, independently, the mathematicial Emil Post. (And in unpublished work of Charles Peirce from about 1890.)

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| $P$ | $Q$ | $P$ and $Q$ |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

# Or

| $P$ | $Q$ | $P$ or $Q$ |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

# Not

| $P$ | not $P$ |
|---|---|
| T | |
| F | |

# If-and-only-if

| $P$ | $Q$ | $P$ iff $Q$ |
|---|---|---|
| T | T | |
| T | F | |
| F | T | |
| F | F | |

# If-then

| $P$ | $Q$ | if $P$ then $Q$ |
|---|---|---|
| T | T | |
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| F | T | |
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# Checking logical equivalence

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For example, let's see that an if-then statement is equvialent to its contrapositive.

| $P$ | $Q$ | if $P$ then $Q$ | if not $Q$ then not $P$ |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

# New logical connectives?

You could introduce new logical connectives and give truth tables for them, but what we have suffices to express anything.

For example, let's see how to express exclusive-or using just 'and' and 'or'.

| $P$ | $Q$ | $P$ xor $Q$ | |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

# DeMorgan's Laws

You can also use truth tables for more complicated logical rules.

DeMorgan's laws state that the following two pairs of statements are equivalent:

- "not ($P$ and $Q$)" and "not $P$ or not $Q$"
- "not ($P$ or $Q$)" and "not $P$ and not $Q$"

| $P$ | $Q$ | not ($P$ and $Q$) | not $P$ or not $Q$ |
|---|---|---|---|
| T | T | | |
| T | F | | |
| F | T | | |
| F | F | | |

# Predicates with parameters

Predicates with parameters are a little trickier to handle.

- Most of the domains of interest to mathematicians are infinite—the natural numbers, the real numbers, and so on.
- So you cannot so easily list all possible combinations of truth values.

# Predicates and sets

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Work over a domain $U$.

- If $P(x)$ is a predicate about objects $x$ from $U$, then the extension of $P(x)$ is the set $\{x \in U : P(x)\}$.
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- Logical connectives applied to predicates correspond to set theoretic operations applied to sets.
  - The extension of "$P(x)$ and $Q(x)$" is the intersection $P \cap Q$.
  - The extension of "$P(x)$ or $Q(x)$" is the union $P \cup Q$.
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  - The extension of "not $P(x)$" is the complement $U \setminus P$.
- You can do similar for if-then and iff, but here's a better way to think about those.
  - "if $P(x)$ then $Q(x)$" is true for all $x \in U$ if and only if $P \subseteq Q$.
  - "$P(x)$ iff $Q(x)$" is true for all $x \in U$ if and only if $P = Q$.

# Venn diagrams

# Quantifiers

- Mathematics is full of statements like "every non-constant polynomial has a root in the complex numbers".
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- It's maybe clearer if we rewrite this statement: "for all nonconstant polynomials $p(x)$ there is a complex number $r$ so that $p(r) = 0$".
  - "For all" is the universal quantifier, talking about all objects of a certain type.
  - "There is" is the existential quantifier, saying there exists some object of a certain type.
- There are symbolic abbreviations:
  - $\forall x\ P(x)$ abbreviates "for all $x$, $P(x)$".
  - $\exists x\ P(x)$ abbreviates "there exists $x$ so that $P(x)$".

# Bounded quantifiers

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When we do this we are bounding the quantifiers.

- Suppose we want to quantify over a smaller domain $D$.
  - "for all $x \in D$, $P(x)$" is equivalent to "for all $x$, if $x \in D$ then $P(x)$".
  - "there exists $x \in D$ so that $P(x)$" is equivalent to "there exists $x$ so that $x \in D$ and $P(x)$".

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  - "there exists $x \in D$ so that $P(x)$" is equivalent to "there exists $x$ so that $x \in D$ and $P(x)$".
- So if you like, you can consider bounded quantifiers an abbreviation. But they're so convenient for phrasing things that we'll use them.

# DeMorgan's laws

Just like we have DeMorgan's laws for and/or, there are DeMorgan's laws for quantifiers. In short, they say that if you push a 'not' inside a quantifier or pull a 'not' out from inside a quantifier, then it flips the quantifier to the other type.

- "not (for all $x$, $P(x)$)" is equivalent to "there exists an $x$ so that not $P(x)$".
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These are often useful for doing proofs by contradictions. For example, if you want to prove that "for all $x$, $P(x)$", you assume it's false—that is, you assume there is some $x$ so that not $P(x)$—and you try to derive a contradiction.