Math 321: Order theory

Kameryn J Williams

University of Hawaiʻi at Mānoa

Spring 2021

Orders

Given a set X, we may be able to compare elements of X, e.g. by size or position.

Let's formalize and abstract this idea.

Orders

Given a set X, we may be able to compare elements of X, e.g. by size or position.

Let's formalize and abstract this idea.

- A binary relation $\leq$ on a set X is an order or partial order if $\leq$ is reflexive, transitive, and antisymmetric.
    - That is, $x \leq x$ for all $x \in X$, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.

Orders

Given a set X, we may be able to compare elements of X, e.g. by size or position.

Let's formalize and abstract this idea.

- A binary relation $\leq$ on a set X is an order or partial order if $\leq$ is reflexive, transitive, and antisymmetric.
  - That is, $x \leq x$ for all $x \in X$, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.
- $<$ is a strict order if it is irreflexive, transitive, and antisymmetric.
  - That is, $x < x$ is never true, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.

Orders

Given a set X, we may be able to compare elements of X, e.g. by size or position.

Let's formalize and abstract this idea.

- A binary relation $\leq$ on a set X is an order or partial order if $\leq$ is reflexive, transitive, and antisymmetric.
  - That is, $x \leq x$ for all $x \in X$, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.
- $<$ is a strict order if it is irreflexive, transitive, and antisymmetric.
  - That is, $x < x$ is never true, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.

Also call orders nonstrict to clearly distinguish.

Orders

Given a set X, we may be able to compare elements of X, e.g. by size or position.

Let's formalize and abstract this idea.

- A binary relation $\leq$ on a set X is an order or partial order if $\leq$ is reflexive, transitive, and antisymmetric.
  - That is, $x \leq x$ for all $x \in X$, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.
- $<$ is a strict order if it is irreflexive, transitive, and antisymmetric.
  - That is, $x < x$ is never true, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.

Also call orders nonstrict to clearly distinguish.

You can transfer from nonstrict to strict orders and vice versa:

- If $\leq$ is a nonstrict order, then $<$ is a strict order, where $x < y$ if $x \leq y$ and $x \neq y$.
- If $\prec$ is a strict order, then $\preceq$ is a nonstrict order, where $x \preceq y$ if $x \prec y$ or $x = y$.

Orders

Given a set X, we may be able to compare elements of X, e.g. by size or position.

Let's formalize and abstract this idea.

- A binary relation $\leq$ on a set X is an order or partial order if $\leq$ is reflexive, transitive, and antisymmetric.
  - That is, $x \leq x$ for all $x \in X$, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.
- $<$ is a strict order if it is irreflexive, transitive, and antisymmetric.
  - That is, $x < x$ is never true, $x \leq y \leq z$ implies $x \leq z$, and $x \leq y$ and $y \leq x$ implies $x = y$.

Also call orders nonstrict to clearly distinguish.

You can transfer from nonstrict to strict orders and vice versa:

- If $\leq$ is a nonstrict order, then $<$ is a strict order, where $x < y$ if $x \leq y$ and $x \neq y$.
- If $\prec$ is a strict order, then $\preceq$ is a nonstrict order, where $x \preceq y$ if $x \prec y$ or $x = y$.

Notation:

- Use symbols like $\leq$, $\preceq$, $\subseteq$, $\sqsubseteq$ to denote orders.
- Write the corresponding strict order as e.g. $<$ or $\lneq$.
- Write it backwards, e.g. $\geq$, to denote the opposite order: $x \geq y$ iff $y \leq x$.

Examples

- $\subseteq$ on $\mathcal{P}(\mathbb{N})$

Examples

- $\subseteq$ on $\mathcal{P}(\mathbb{N})$
- $\leq$ on $\mathbb{R}$

Examples

- $\subseteq$ on $\mathcal{P}(\mathbb{N})$
- $\leq$ on $\mathbb{R}$
- $\mid$ on $\mathbb{N}$

Examples

- $\subseteq$ on $\mathcal{P}(\mathbb{N})$
- $\leq$ on $\mathbb{R}$
- $\mid$ on $\mathbb{N}$
- The subsequence relation $\sqsubseteq$ on $\mathbb{N}^*$

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

- Any least element is minimal.

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

- Any least element is minimal.

If there were $x < \ell$ then it would be false that $\ell \leq x$,

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

- Any least element is minimal.

If there were $x < \ell$ then it would be false that $\ell \leq x$, because then we would have $\ell \leq x < \ell$ and so $\ell < \ell$ and so $\ell \neq \ell$.

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

- Any least element is minimal.

If there were $x < \ell$ then it would be false that $\ell \leq x$, because then we would have $\ell \leq x < \ell$ and so $\ell < \ell$ and so $\ell \neq \ell$.

- If ℓ is a least element it is the only minimal element, and thus also the only least element.

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

- Any least element is minimal.

If there were $x < \ell$ then it would be false that $\ell \leq x$, because then we would have $\ell \leq x < \ell$ and so $\ell < \ell$ and so $\ell \neq \ell$.

- If ℓ is a least element it is the only minimal element, and thus also the only least element.

Consider $m \in X$. Because $\ell \leq m$ the only way m can be minimal is if $m = \ell$.

Minimal and least elements

Consider an ordered set $(X, <)$.

- $m \in X$ is a minimal element if there is nothing smaller than it. In symbols: if there is no $x \in X$ with $x < m$.
- $\ell \in X$ is a least element if it is smaller than everything else. In symbols: if $\ell \leq x$ for all $x \in X$.

The dual notions are maximal and greatest elements.

- $M \in X$ is a maximal element if there is nothing larger than it.
- $G \in X$ is a greatest element if it is larger than everything else.

These are minimal/least in the opposite order.

- Any least element is minimal.

If there were $x < \ell$ then it would be false that $\ell \leq x$, because then we would have $\ell \leq x < \ell$ and so $\ell < \ell$ and so $\ell \neq \ell$.

- If ℓ is a least element it is the only minimal element, and thus also the only least element.

Consider $m \in X$. Because $\ell \leq m$ the only way m can be minimal is if $m = \ell$.

- Being the unique minimal element doesn't imply being least.

A unique minimal element which isn't least

Linear orders

An order $\leq$ on a set X is a linear order or total order if it satisfies the trichotomy property:

- For any points $x, y \in X$, either $x < y$, $x = y$, or $x > y$.

Linear orders

An order $\leq$ on a set X is a linear order or total order if it satisfies the trichotomy property:

- For any points $x, y \in X$, either $x < y$, $x = y$, or $x > y$.

- In a linear order, an element is least iff it is minimal.

Linear orders

An order $\leq$ on a set X is a linear order or total order if it satisfies the trichotomy property:

- For any points $x, y \in X$, either $x < y$, $x = y$, or $x > y$.

- In a linear order, an element is least iff it is minimal.

We already saw least $\Rightarrow$ minimal in a more general context, so let's see the other direction.

Linear orders

An order $\leq$ on a set X is a linear order or total order if it satisfies the trichotomy property:

- For any points $x, y \in X$, either $x < y$, $x = y$, or $x > y$.

- In a linear order, an element is least iff it is minimal.

We already saw least $\Rightarrow$ minimal in a more general context, so let's see the other direction. Suppose m is minimal, consider $x \in X$. By trichotomy, either $x < m$ or $m \leq x$. We know $x < m$ cannot be, so it must be $m \leq x$.

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Outside of maths, pre-orders get used for preference theory, where $a \lesssim b$ means "b is at least as preferable as a".

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Outside of maths, pre-orders get used for preference theory, where $a \lesssim b$ means "b is at least as preferable as a".

- For example, my preferences for ice cream might include: chocolate $\lesssim$ mint $\lesssim$ pistachio $\lesssim$ chocolate.

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Outside of maths, pre-orders get used for preference theory, where $a \lesssim b$ means "b is at least as preferable as a".

- For example, my preferences for ice cream might include: chocolate $\lesssim$ mint $\lesssim$ pistachio $\lesssim$ chocolate.

It seems like I'm saying I like these all the same, and indeed we can formalize this.

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Outside of maths, pre-orders get used for preference theory, where $a \lesssim b$ means "b is at least as preferable as a".

- For example, my preferences for ice cream might include: chocolate $\lesssim$ mint $\lesssim$ pistachio $\lesssim$ chocolate.

It seems like I'm saying I like these all the same, and indeed we can formalize this.

- Given a pre-order $\lesssim$, define a relation $\sim$ as $x \sim y$ if $x \lesssim y \lesssim x$. This is an equivalence relation.

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Outside of maths, pre-orders get used for preference theory, where $a \lesssim b$ means "b is at least as preferable as a".

- For example, my preferences for ice cream might include: chocolate $\lesssim$ mint $\lesssim$ pistachio $\lesssim$ chocolate.

It seems like I'm saying I like these all the same, and indeed we can formalize this.

- Given a pre-order $\lesssim$, define a relation $\sim$ as $x \sim y$ if $x \lesssim y \lesssim x$. This is an equivalence relation.
- Moreover, the relation $\leq$ defined on the $\sim$-equivalence classes as $[x] \leq [y]$ if $x \lesssim y$ is well-defined and is a partial order.

Pre-orders

A relation $\lesssim$ on a set X is a pre-order if it is reflexive and transitive.

- That is, it's like an order except we don't require anti-symmetry.

Outside of maths, pre-orders get used for preference theory, where $a \lesssim b$ means "b is at least as preferable as a".

- For example, my preferences for ice cream might include: chocolate $\lesssim$ mint $\lesssim$ pistachio $\lesssim$ chocolate.

It seems like I'm saying I like these all the same, and indeed we can formalize this.

- Given a pre-order $\lesssim$, define a relation $\sim$ as $x \sim y$ if $x \lesssim y \lesssim x$. This is an equivalence relation.
- Moreover, the relation $\leq$ defined on the $\sim$-equivalence classes as $[x] \leq [y]$ if $x \lesssim y$ is well-defined and is a partial order.

For example, we compared sizes of sets: $A \lesssim B$ if there is a one-to-one function from A to B and $A \approx B$ if there is a bijection from A to B. The Cantor–Schröder–Bernstein theorem says $A \approx B$ iff $A \lesssim B \lesssim A$. Going from this pre-order to the order on the equivalence classes goes from comparing individual sets to comparing sizes of sets.

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

Can we get a general notion of when two orders are the same?

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

Can we get a general notion of when two orders are the same?

Let $(X, <_X)$ and $(Y, <_Y)$ be orders.

- An isomorphsim from X to Y is a bijection $\pi : X \to Y$ so that $a <_X b$ iff $\pi(a) <_Y \pi(b)$ for all $a, b \in X$.
- Two orders are isomorphic if there is an isomorphism between them.

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

Can we get a general notion of when two orders are the same?

Let $(X, <_X)$ and $(Y, <_Y)$ be orders.

- An isomorphsim from X to Y is a bijection $\pi : X \to Y$ so that $a <_X b$ iff $\pi(a) <_Y \pi(b)$ for all $a, b \in X$.
- Two orders are isomorphic if there is an isomorphism between them.

Some examples:

- $\mathbb{N}$ versus $\{\text{evens}\}$

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

Can we get a general notion of when two orders are the same?

Let $(X, <_X)$ and $(Y, <_Y)$ be orders.

- An isomorphsim from X to Y is a bijection $\pi : X \to Y$ so that $a <_X b$ iff $\pi(a) <_Y \pi(b)$ for all $a, b \in X$.
- Two orders are isomorphic if there is an isomorphism between them.

Some examples:

- $\mathbb{N}$ versus $\{\text{evens}\}$
- $\mathbb{R}$ versus $(-\pi/2, \pi/2)$

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

Can we get a general notion of when two orders are the same?

Let $(X, <_X)$ and $(Y, <_Y)$ be orders.

- An isomorphsim from X to Y is a bijection $\pi : X \to Y$ so that $a <_X b$ iff $\pi(a) <_Y \pi(b)$ for all $a, b \in X$.
- Two orders are isomorphic if there is an isomorphism between them.

Some examples:

- $\mathbb{N}$ versus $\{\text{evens}\}$
- $\mathbb{R}$ versus $(-\pi/2, \pi/2)$
- $[0, 1]$ versus $[0, 2]$

When are two orders the same?

Consider the following two orders:

- $A < B < C$
- $1 < 2 < 3$

Clearly these are the same, I just relabeled the elements with new names.

Can we get a general notion of when two orders are the same?

Let $(X, <_X)$ and $(Y, <_Y)$ be orders.

- An isomorphsim from X to Y is a bijection $\pi : X \to Y$ so that $a <_X b$ iff $\pi(a) <_Y \pi(b)$ for all $a, b \in X$.
- Two orders are isomorphic if there is an isomorphism between them.

Some examples:

- $\mathbb{N}$ versus $\{\text{evens}\}$
- $\mathbb{R}$ versus $(-\pi/2, \pi/2)$
- $[0, 1]$ versus $[0, 2]$

Observe that the isomorphisms here don't preserve any structure beyond the order information—e.g. algebraic information is lost.

Given any notion of a mathematical structure, there's a corresponding notion of isomorphism—a bijection which preserves all the structure.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $\mid$ are isomorphic.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $\mid$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $|$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $\mid$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

- Claim: For $n, m \in S$, $n \mid m$ iff $\pi(n) \subseteq \pi(m)$.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $\mid$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

- Claim: For $n, m \in S$, $n \mid m$ iff $\pi(n) \subseteq \pi(m)$.

n divides m iff each prime in n's prime factorization appear in m's prime factorization, but that's just saying $\pi(n) \subseteq \pi(m)$.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $|$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

- Claim: For $n, m \in S$, $n \mid m$ iff $\pi(n) \subseteq \pi(m)$.

n divides m iff each prime in n's prime factorization appear in m's prime factorization, but that's just saying $\pi(n) \subseteq \pi(m)$.

- Claim: π is onto.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $|$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

- Claim: For $n, m \in S$, $n \mid m$ iff $\pi(n) \subseteq \pi(m)$.

n divides m iff each prime in n's prime factorization appear in m's prime factorization, but that's just saying $\pi(n) \subseteq \pi(m)$.

- Claim: π is onto.

Let A be a finite set of primes. If n is the product of the primes in A then $\pi(n) = A$.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $\mid$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

- Claim: For $n, m \in S$, $n \mid m$ iff $\pi(n) \subseteq \pi(m)$.

n divides m iff each prime in n's prime factorization appear in m's prime factorization, but that's just saying $\pi(n) \subseteq \pi(m)$.

- Claim: π is onto.

Let A be a finite set of primes. If n is the product of the primes in A then $\pi(n) = A$.

- Claim: $\mathcal{P}_{\text{fin}}(P)$ is isomorphic to $\mathcal{P}_{\text{fin}}(\mathbb{N})$.

A more difficult example of isomorphism

- $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of *finite* sets of natural numbers.
- S denotes the set of positive square-free integers, those whose prime factorizations only have 1 as an exponent.

Theorem

$\mathcal{P}_{\text{fin}}(\mathbb{N})$ under $\subseteq$ and S under divisibility $|$ are isomorphic.

Here's the idea: given $n \in S$, associate to n its set of prime factors. Because n is square-free, this is a one-to-one map from S to $\mathcal{P}_{\text{fin}}(P)$, the set of finite sets of primes.

Call this correspondence $\pi : S \to \mathcal{P}_{\text{fin}}(P)$.

- Claim: For $n, m \in S$, $n \mid m$ iff $\pi(n) \subseteq \pi(m)$.

n divides m iff each prime in n's prime factorization appear in m's prime factorization, but that's just saying $\pi(n) \subseteq \pi(m)$.

- Claim: π is onto.

Let A be a finite set of primes. If n is the product of the primes in A then $\pi(n) = A$.

- Claim: $\mathcal{P}_{\text{fin}}(P)$ is isomorphic to $\mathcal{P}_{\text{fin}}(\mathbb{N})$.

Any bijection $P \to \mathbb{N}$ gives rise to an isomorphism $\mathcal{P}_{\text{fin}}(P) \to \mathcal{P}_{\text{fin}}(\mathbb{N})$.

Another isomorphism result

Consider a linear order $(X, <)$.

- X is dense if given any $x < z$ from X there $y \in X$ with $x < y < z$.
- X is endless if has neither a maximum nor a minimum.

Another isomorphism result

Consider a linear order $(X, <)$.

- X is dense if given any $x < z$ from X there $y \in X$ with $x < y < z$.
- X is endless if has neither a maximum nor a minimum.

Theorem (Cantor)

Any two countable endless dense linear orders are isomorphic. In particular, every countable endless dense linear order is isomorphic to $(\mathbb{Q}, <)$.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

Enumerate X and Y as, respectively

- $x_0, x_1, \ldots, x_n, \ldots$
- $y_0, y_1, \ldots, y_m, \ldots$

We inductively build an isomorphism π.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

Enumerate X and Y as, respectively

- $x_0, x_1, \ldots, x_n, \ldots$
- $y_0, y_1, \ldots, y_m, \ldots$

We inductively build an isomorphism π.

(Base case) Set $\pi(x_0) = y_0$.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

Enumerate X and Y as, respectively

- $x_0, x_1, \ldots, x_n, \ldots$
- $y_0, y_1, \ldots, y_m, \ldots$

We inductively build an isomorphism π.

(Base case) Set $\pi(x_0) = y_0$.

(Successor step) We do a forth and a back step, extending the finite piece of π we have built so far.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

Enumerate X and Y as, respectively

- $x_0, x_1, \ldots, x_n, \ldots$
- $y_0, y_1, \ldots, y_m, \ldots$

We inductively build an isomorphism π.

(Base case) Set $\pi(x_0) = y_0$.

(Successor step) We do a forth and a back step, extending the finite piece of π we have built so far.

(Forth) Look at the first x_n, according to the enumeration, which we haven't handled yet. Map it to the first y_m which fits in with the $\pi(x_i)$ the same as x_n fits in with the x_i we've already handled.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

Enumerate X and Y as, respectively

- $x_0, x_1, \ldots, x_n, \ldots$
- $y_0, y_1, \ldots, y_m, \ldots$

We inductively build an isomorphism π.

(Base case) Set $\pi(x_0) = y_0$.

(Successor step) We do a forth and a back step, extending the finite piece of π we have built so far.

(Forth) Look at the first x_n, according to the enumeration, which we haven't handled yet. Map it to the first y_m which fits in with the $\pi(x_i)$ the same as x_n fits in with the x_i we've already handled.

(Back) Look at the first y_m which we haven't handled yet. Look for the first x_n which fits in with the x_i we've already handled the same as y_m fits in wiith the $\pi(x_i)$. Set $\pi(x_n) = y_m$.

A back-and-forth proof for Cantor's theorem

- Any two conutable endless dense linear orders are isomorphic.

Suppose $(X, <_X)$ and $(Y, <_Y)$ are countable EDLOs.

Enumerate X and Y as, respectively

- $x_0, x_1, \ldots, x_n, \ldots$
- $y_0, y_1, \ldots, y_m, \ldots$

We inductively build an isomorphism π.

(Base case) Set $\pi(x_0) = y_0$.

(Successor step) We do a forth and a back step, extending the finite piece of π we have built so far.

(Forth) Look at the first x_n, according to the enumeration, which we haven't handled yet. Map it to the first y_m which fits in with the $\pi(x_i)$ the same as x_n fits in with the x_i we've already handled.

(Back) Look at the first y_m which we haven't handled yet. Look for the first x_n which fits in with the x_i we've already handled the same as y_m fits in wiith the $\pi(x_i)$. Set $\pi(x_n) = y_m$.

After countable many steps we've built π. By construction, π is injective and preserves the order. The forth step ensures $\operatorname{dom} \pi = X$. And the back step ensures $\operatorname{ran} \pi = Y$. So π is the desired isomorphism.

A related theorem

Theorem (Cantor)

Every countable linear order embeds into $(\mathbb{Q}, <)$. That is, if $(X, <_X)$ is a countable linear order then $(X, <_X)$ is isomorphic to a suborder of $(\mathbb{Q}, <)$.

A related theorem

Theorem (Cantor)

Every countable linear order embeds into $(\mathbb{Q}, <)$. That is, if $(X, <_X)$ is a countable linear order then $(X, <_X)$ is isomorphic to a suborder of $(\mathbb{Q}, <)$.

Proof sketch: Like the back-and-forth argument, except we only need the forth step to build an embedding $\pi : X \to \mathbb{Q}$.

See page 169 in the textbook for the full proof.