Math 321: Relations and functions

Kameryn J Williams

University of Hawaiʻi at Mānoa

Spring 2021

Objects in mathematics

In mathematics, we are interested in many kinds of objects

- Numbers, sets, graphs, trees, . . .

Objects in mathematics

In mathematics, we are interested in many kinds of objects

- Numbers, sets, graphs, trees, ...

We don't just want to talk about individual objects, we want to talk about what we can do with them and how they relate

- $2 + 3 > 4 - 1$
- $\mathbb{N} \subseteq \mathbb{R}$
- 18 is a multiple of 2

Objects in mathematics

In mathematics, we are interested in many kinds of objects

- Numbers, sets, graphs, trees, ...

We don't just want to talk about individual objects, we want to talk about what we can do with them and how they relate

- $2 + 3 > 4 - 1$
- $\mathbb{N} \subseteq \mathbb{R}$
- 18 is a multiple of 2

These functions and relations are themselves objects of mathematical study.

Relations

Consider the relation "a divides b", where a and b are natural numbers. Let's write this as

$$a \mid b.$$

Relations

Consider the relation "a divides b", where a and b are natural numbers. Let's write this as

$$a \mid b.$$

One way we can think about this relation is in terms of its definition: $a \mid b$ iff there is an integer k so that $ak = b$. This is the intension of the relation, telling us under what conditions two objects are related in this way.

Relations

Consider the relation "a divides b", where a and b are natural numbers. Let's write this as

$$a \mid b.$$

One way we can think about this relation is in terms of its definition: $a \mid b$ iff there is an integer k so that $ak = b$. This is the intension of the relation, telling us under what conditions two objects are related in this way.

Another way we can think of it is in terms of its extension, the information about which objects are related in this way. This amounts to talking about the set of ordered pairs $(a, b) \in \mathbb{N}^2$ so that $a \mid b$.

Relations

Consider the relation "a divides b", where a and b are natural numbers. Let's write this as

$$a \mid b.$$

One way we can think about this relation is in terms of its definition: $a \mid b$ iff there is an integer k so that $ak = b$. This is the intension of the relation, telling us under what conditions two objects are related in this way.

Another way we can think of it is in terms of its extension, the information about which objects are related in this way. This amounts to talking about the set of ordered pairs $(a, b) \in \mathbb{N}^2$ so that $a \mid b$.

The standard in mathematics is to take the extension of a relation as *the* definition of relations.

- A binary relation on a set A is a subset of A^2, the set of ordered pairs from A.
- If $\star$ is a relation on A, then $a \star b$ just means $(a, b) \in \star$.

Relations

Consider the relation "a divides b", where a and b are natural numbers. Let's write this as

$$a \mid b.$$

One way we can think about this relation is in terms of its definition: $a \mid b$ iff there is an integer k so that $ak = b$. This is the intension of the relation, telling us under what conditions two objects are related in this way.

Another way we can think of it is in terms of its extension, the information about which objects are related in this way. This amounts to talking about the set of ordered pairs $(a, b) \in \mathbb{N}^2$ so that $a \mid b$.

The standard in mathematics is to take the extension of a relation as *the* definition of relations.

- A binary relation on a set A is a subset of A^2, the set of ordered pairs from A.
- If $\star$ is a relation on A, then $a \star b$ just means $(a, b) \in \star$.

(You can talk about relations between more than two objects, but binary relations are used the most.)

Some examples

- $<$ on $\mathbb{R}$

Some examples

- $<$ on $\mathbb{R}$
- $\leq$ on $\mathbb{R}$

Some examples

- $<$ on $\mathbb{R}$
- $\leq$ on $\mathbb{R}$
- $=$ on $\mathbb{R}$

Some examples

- $<$ on $\mathbb{R}$
- $\leq$ on $\mathbb{R}$
- $=$ on $\mathbb{R}$
- $\neq$ on $\mathbb{R}$

Some examples

- $<$ on $\mathbb{R}$
- $\leq$ on $\mathbb{R}$
- $=$ on $\mathbb{R}$
- $\neq$ on $\mathbb{R}$
- $\subseteq$ on $\mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$

Some examples

- $<$ on $\mathbb{R}$
- $\leq$ on $\mathbb{R}$
- $=$ on $\mathbb{R}$
- $\neq$ on $\mathbb{R}$
- $\subseteq$ on $\mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$
- $\equiv \mod 3$ on $\mathbb{Z}$ (that is, a and b have the same remainder when divided by 3)
  This is called equivalence modulo 3.

Some properties a relation can have

Let $\star$ be a binary relation on A.

- $\star$ is reflexive if $a \star a$ for all $a \in A$.
- $\star$ is symmetric if $a \star b$ implies $b \star a$ for all $a, b \in A$.
- $\star$ is transitive if $a \star b$ and $b \star c$ implies $a \star c$ for all $a, b, c \in A$.

Some properties a relation can have

Let $\star$ be a binary relation on A.

- $\star$ is reflexive if $a \star a$ for all $a \in A$.
- $\star$ is symmetric if $a \star b$ implies $b \star a$ for all $a, b \in A$.
- $\star$ is transitive if $a \star b$ and $b \star c$ implies $a \star c$ for all $a, b, c \in A$.

- $<$ on $\mathbb{R}$
- $\leq$ on $\mathbb{R}$
- $=$ on $\mathbb{R}$
- $\neq$ on $\mathbb{R}$
- $\subseteq$ on $\mathcal{P}(\mathbb{N})$
- $\equiv \pmod 3$ on $\mathbb{Z}$

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

This theorem is giving existential statements, so to prove it we just have to provide an example for each of the eight cases.

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

This theorem is giving existential statements, so to prove it we just have to provide an example for each of the eight cases.

- (RST) We checked earlier that $=$ on $\mathbb{R}$ has all three properties.
- (RT) We checked earlier that $\leq$ on $\mathbb{R}$ is reflexive and transitive but not symmetric.
- (T) We checked earlier that $<$ on $\mathbb{R}$ is transitive but neither reflexive nor symmetric.
- (S) We checked earlier that $\neq$ on $\mathbb{R}$ is symmetric but neither reflexive nor transitive.

So we just need to provide examples for the remaining four cases.

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

This theorem is giving existential statements, so to prove it we just have to provide an example for each of the eight cases.

- (RST) We checked earlier that $=$ on $\mathbb{R}$ has all three properties.
- (RT) We checked earlier that $\leq$ on $\mathbb{R}$ is reflexive and transitive but not symmetric.
- (T) We checked earlier that $<$ on $\mathbb{R}$ is transitive but neither reflexive nor symmetric.
- (S) We checked earlier that $\neq$ on $\mathbb{R}$ is symmetric but neither reflexive nor transitive.

So we just need to provide examples for the remaining four cases.

- ($\emptyset$)
- (R)
- (RS)
- (ST)

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

- ($\emptyset$) Consider the relation $\star$ on $\mathbb{N}$ defined as $a \star b$ iff $a = 0$ and $b = 1$. This relation is neither reflexive, symmetric, nor transitive.

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

- ($\emptyset$) Consider the relation $\star$ on $\mathbb{N}$ defined as $a \star b$ iff $a = 0$ and $b = 1$. This relation is neither reflexive, symmetric, nor transitive.
- (R) Consider the relation $*$ on $\mathbb{Z}$ defined as $a * b$ iff either $b = a$ or $b = a + 1$. This is reflexive but neither symmetric nor transitive.

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

- ($\emptyset$) Consider the relation $\star$ on $\mathbb{N}$ defined as $a \star b$ iff $a = 0$ and $b = 1$. This relation is neither reflexive, symmetric, nor transitive.
- (R) Consider the relation $*$ on $\mathbb{Z}$ defined as $a * b$ iff either $b = a$ or $b = a + 1$. This is reflexive but neither symmetric nor transitive.
- (RS) Consider the relation $\dagger$ on $\mathbb{Z}$ defined as $a \dagger b$ iff $|a - b| \leq 1$. This relation is reflexive and symmetric but not transitive.

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

- ($\emptyset$) Consider the relation $\star$ on $\mathbb{N}$ defined as $a \star b$ iff $a = 0$ and $b = 1$. This relation is neither reflexive, symmetric, nor transitive.
- (R) Consider the relation $*$ on $\mathbb{Z}$ defined as $a * b$ iff either $b = a$ or $b = a + 1$. This is reflexive but neither symmetric nor transitive.
- (RS) Consider the relation $\dagger$ on $\mathbb{Z}$ defined as $a \dagger b$ iff $|a - b| \leq 1$. This relation is reflexive and symmetric but not transitive.
- (ST) Consider the empty relation on a (nonempty) set A: a and b are *never* related. This relation is vacuously symmetric and transitive, but not reflexive.

Reflexivity, symmetry, and transitivity

Theorem

No combination of reflexivity, symmetry, and transitive implies the other. For each of the eight possible combinations of which properties hold, there is a binary relation which has exactly those properties.

- ($\emptyset$) Consider the relation $\star$ on $\mathbb{N}$ defined as $a \star b$ iff $a = 0$ and $b = 1$. This relation is neither reflexive, symmetric, nor transitive.
- (R) Consider the relation $*$ on $\mathbb{Z}$ defined as $a * b$ iff either $b = a$ or $b = a + 1$. This is reflexive but neither symmetric nor transitive.
- (RS) Consider the relation $\dagger$ on $\mathbb{Z}$ defined as $a \dagger b$ iff $|a - b| \leq 1$. This relation is reflexive and symmetric but not transitive.
- (ST) Consider the empty relation on a (nonempty) set A: a and b are *never* related. This relation is vacuously symmetric and transitive, but not reflexive.

We have examples for all eight cases, completing the proof of the theorem.

Closures of a relation

Let $\star$ be a relation on a set A. We can add new instances to $\star$ to make it satisfy these properties.

- The reflexive closure of $\star$ is the smallest reflexive relation on A which contains $\star$, i.e. as a subset.
- The symmetric closure of $\star$ is the smallest symmetric relation on A which contains $\star$.
- The transitive closure of $\star$ is the smallest transitive relation on A which contains $\star$.

The reflexive symmetric, reflexive transitive, symmetric transitive, and reflexive symmetric transitive closures are defined similarly, in the obvious manner.

Closures of a relation

Let $\star$ be a relation on a set A. We can add new instances to $\star$ to make it satisfy these properties.

- The reflexive closure of $\star$ is the smallest reflexive relation on A which contains $\star$, i.e. as a subset.
- The symmetric closure of $\star$ is the smallest symmetric relation on A which contains $\star$.
- The transitive closure of $\star$ is the smallest transitive relation on A which contains $\star$.

The reflexive symmetric, reflexive transitive, symmetric transitive, and reflexive symmetric transitive closures are defined similarly, in the obvious manner.

Examples:

- What is the reflexive closure of $<$ on $\mathbb{R}$?
- What is the symmetric closure of $<$ on $\mathbb{R}$?
- What is the transitive closure of $<$ on $\mathbb{R}$?

Closures of a relation

Let $\star$ be a relation on a set A. We can add new instances to $\star$ to make it satisfy these properties.

- The reflexive closure of $\star$ is the smallest reflexive relation on A which contains $\star$, i.e. as a subset.
- The symmetric closure of $\star$ is the smallest symmetric relation on A which contains $\star$.
- The transitive closure of $\star$ is the smallest transitive relation on A which contains $\star$.

The reflexive symmetric, reflexive transitive, symmetric transitive, and reflexive symmetric transitive closures are defined similarly, in the obvious manner.

Examples:

- What is the reflexive closure of $<$ on $\mathbb{R}$?
- What is the symmetric closure of $<$ on $\mathbb{R}$?
- What is the transitive closure of $<$ on $\mathbb{R}$?
- What is the reflexive closure of $\neq$ on $\mathbb{R}$?
- What is the transitive closure of $\neq$ on $\mathbb{R}$?

Closures of a relation

Let $\star$ be a relation on a set A. We can add new instances to $\star$ to make it satisfy these properties.

- The reflexive closure of $\star$ is the smallest reflexive relation on A which contains $\star$, i.e. as a subset.
- The symmetric closure of $\star$ is the smallest symmetric relation on A which contains $\star$.
- The transitive closure of $\star$ is the smallest transitive relation on A which contains $\star$.

The reflexive symmetric, reflexive transitive, symmetric transitive, and reflexive symmetric transitive closures are defined similarly, in the obvious manner.

Examples:

- What is the reflexive closure of $<$ on $\mathbb{R}$?
- What is the symmetric closure of $<$ on $\mathbb{R}$?
- What is the transitive closure of $<$ on $\mathbb{R}$?
- What is the reflexive closure of $\neq$ on $\mathbb{R}$?
- What is the transitive closure of $\neq$ on $\mathbb{R}$?
- What is the reflexive closure of the empty relation on $\mathbb{R}$?

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.
- The symmetric closure of $\dagger$ is $\dagger \cup \rotatebox[origin=c]{180}{$\dagger$}$, where $\rotatebox[origin=c]{180}{$\dagger$}$ is the opposite relation to $\dagger$. Namely, $\rotatebox[origin=c]{180}{$\dagger$}$ is defined as $a \rotatebox[origin=c]{180}{$\dagger$} b$ iff $b \dagger a$.

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.
- The symmetric closure of $\dagger$ is $\dagger \cup$ ⸸, where ⸸ is the opposite relation to $\dagger$. Namely, ⸸ is defined as a ⸸ b iff $b \dagger a$.
- Can you describe the reflexive symmetric closure of $\dagger$?

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.
- The symmetric closure of $\dagger$ is $\dagger \cup$ ⸸, where ⸸ is the opposite relation to $\dagger$. Namely, ⸸ is defined as a ⸸ b iff $b \dagger a$.
- Can you describe the reflexive symmetric closure of $\dagger$?

The transitive closure is a trickier. Trying to close off in one step creates new issues to fix.

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.
- The symmetric closure of $\dagger$ is $\dagger \cup$ ⸸, where ⸸ is the opposite relation to $\dagger$. Namely, ⸸ is defined as a ⸸ b iff $b \dagger a$.
- Can you describe the reflexive symmetric closure of $\dagger$?

The transitive closure is a trickier. Trying to close off in one step creates new issues to fix.

Consider the differ by ≤ 1 relation on $\mathbb{Z}$: $a \dagger b$ iff $|a - b| \leq 1$.

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.
- The symmetric closure of $\dagger$ is $\dagger \cup \rotatebox[origin=c]{180}{$\dagger$}$, where $\rotatebox[origin=c]{180}{$\dagger$}$ is the opposite relation to $\dagger$. Namely, $\rotatebox[origin=c]{180}{$\dagger$}$ is defined as $a \mathrel{\rotatebox[origin=c]{180}{$\dagger$}} b$ iff $b \dagger a$.
- Can you describe the reflexive symmetric closure of $\dagger$?

The transitive closure is a trickier. Trying to close off in one step creates new issues to fix.

Consider the differ by ≤ 1 relation on $\mathbb{Z}$: $a \dagger b$ iff $|a - b| \leq 1$.

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

Another way to define closures

Let $\dagger$ be a binary relation on a set A.

- The reflexive closure of $\dagger$ is $\dagger \cup =$. That is, we add in all pairs of the form (a, a) to $\star$ to get its reflexive closure.
- The symmetric closure of $\dagger$ is $\dagger \cup \rotatebox[origin=c]{180}{$\dagger$}$, where $\rotatebox[origin=c]{180}{$\dagger$}$ is the opposite relation to $\dagger$. Namely, $\rotatebox[origin=c]{180}{$\dagger$}$ is defined as $a \rotatebox[origin=c]{180}{$\dagger$} b$ iff $b \dagger a$.
- Can you describe the reflexive symmetric closure of $\dagger$?

The transitive closure is a trickier. Trying to close off in one step creates new issues to fix.

Consider the differ by ≤ 1 relation on $\mathbb{Z}$: $a \dagger b$ iff $|a - b| \leq 1$.

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\begin{aligned}\bar{\dagger} &= \bigcup_{n=0}^{\infty} \dagger_n \\ &= \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots\end{aligned}$$

To see this is really the transitive closure, we have to check three things: (1) $\bar{\dagger}$ contains $\dagger$, (2) $\bar{\dagger}$ is transitive, and (3) any transitive relation which contains $\dagger$ also contains $\bar{\dagger}$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\begin{aligned} \bar{\dagger} &= \bigcup_{n=0}^{\infty} \dagger_n \\ &= \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots \end{aligned}$$

To see this is really the transitive closure, we have to check three things: (1) $\bar{\dagger}$ contains $\dagger$, (2) $\bar{\dagger}$ is transitive, and (3) any transitive relation which contains $\dagger$ also contains $\bar{\dagger}$.

- ($\bar{\dagger}$ contains $\dagger$) True by construction.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

To see this is really the transitive closure, we have to check three things: (1) $\bar{\dagger}$ contains $\dagger$, (2) $\bar{\dagger}$ is transitive, and (3) any transitive relation which contains $\dagger$ also contains $\bar{\dagger}$.

- ($\bar{\dagger}$ contains $\dagger$) True by construction.
- ($\bar{\dagger}$ is transitive) Suppose $a \,\bar{\dagger}\, b \,\bar{\dagger}\, c$. We want to see $a \,\bar{\dagger}\, c$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

To see this is really the transitive closure, we have to check three things: (1) $\bar{\dagger}$ contains $\dagger$, (2) $\bar{\dagger}$ is transitive, and (3) any transitive relation which contains $\dagger$ also contains $\bar{\dagger}$.

- ($\bar{\dagger}$ contains $\dagger$) True by construction.
- ($\bar{\dagger}$ is transitive) Suppose $a \,\bar{\dagger}\, b \,\bar{\dagger}\, c$. We want to see $a \,\bar{\dagger}\, c$. By definition of $\bar{\dagger}$, there is large enough n so that $a \dagger_n b \dagger_n c$. So then $a \dagger_{n+1} c$, and thus $a \,\bar{\dagger}\, c$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\overline{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

To see this is really the transitive closure, we have to check three things: (1) $\overline{\dagger}$ contains $\dagger$, (2) $\overline{\dagger}$ is transitive, and (3) any transitive relation which contains $\dagger$ also contains $\overline{\dagger}$.

- ($\overline{\dagger}$ contains $\dagger$) True by construction.
- ($\overline{\dagger}$ is transitive) Suppose $a \mathrel{\overline{\dagger}} b \mathrel{\overline{\dagger}} c$. We want to see $a \mathrel{\overline{\dagger}} c$. By definition of $\overline{\dagger}$, there is large enough n so that $a \dagger_n b \dagger_n c$. So then $a \dagger_{n+1} c$, and thus $a \mathrel{\overline{\dagger}} c$.
- ($\overline{\dagger}$ is contained in any transitive relation which contains $\dagger$) Observe that, by the definition of $\overline{\dagger}$, it is enough to prove that if a transitive relation extends $\dagger$ it contains each $\dagger_n$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

To see this is really the transitive closure, we have to check three things: (1) $\bar{\dagger}$ contains $\dagger$, (2) $\bar{\dagger}$ is transitive, and (3) any transitive relation which contains $\dagger$ also contains $\bar{\dagger}$.

- ($\bar{\dagger}$ contains $\dagger$) True by construction.
- ($\bar{\dagger}$ is transitive) Suppose $a \,\bar{\dagger}\, b \,\bar{\dagger}\, c$. We want to see $a \,\bar{\dagger}\, c$. By definition of $\bar{\dagger}$, there is large enough n so that $a \dagger_n b \dagger_n c$. So then $a \dagger_{n+1} c$, and thus $a \,\bar{\dagger}\, c$.
- ($\bar{\dagger}$ is contained in any transitive relation which contains $\dagger$) Observe that, by the definition of $\bar{\dagger}$, it is enough to prove that if a transitive relation extends $\dagger$ it contains each $\dagger_n$. Let's do this induction.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

- (Base case) $\star$ contains $\dagger_0$ because $\dagger_0$ is just $\dagger$, and this is true by assumption.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

- (Base case) $\star$ contains $\dagger_0$ because $\dagger_0$ is just $\dagger$, and this is true by assumption.
- (Inductive step) Assume that $\star$ contains $\dagger_n$. We want to see that if $a \dagger_{n+1} c$ then $a \star c$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

- (Base case) $\star$ contains $\dagger_0$ because $\dagger_0$ is just $\dagger$, and this is true by assumption.
- (Inductive step) Assume that $\star$ contains $\dagger_n$. We want to see that if $a \dagger_{n+1} c$ then $a \star c$. By definition, $a \dagger_{n+1} c$ if either $a \dagger_n c$ or there is b so that $a \dagger_n b \dagger_n c$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a, c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

- (Base case) $\star$ contains $\dagger_0$ because $\dagger_0$ is just $\dagger$, and this is true by assumption.
- (Inductive step) Assume that $\star$ contains $\dagger_n$. We want to see that if $a \dagger_{n+1} c$ then $a \star c$. By definition, $a \dagger_{n+1} c$ if either $a \dagger_n c$ or there is b so that $a \dagger_n b \dagger_n c$.
  - Case 1 ($a \dagger_n c$): Then $a \star c$ by inductive hypothesis.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

- (Base case) $\star$ contains $\dagger_0$ because $\dagger_0$ is just $\dagger$, and this is true by assumption.
- (Inductive step) Assume that $\star$ contains $\dagger_n$. We want to see that if $a \dagger_{n+1} c$ then $a \star c$. By definition, $a \dagger_{n+1} c$ if either $a \dagger_n c$ or there is b so that $a \dagger_n b \dagger_n c$.
  - Case 1 ($a \dagger_n c$): Then $a \star c$ by inductive hypothesis.
  - Case 2 ($a \dagger_n b \dagger_n c$): By inductive hypothesis, $a \star b \star c$. Because $\star$ is transitive, we conclude $a \star c$.

Looking more closely at transitive closures

Instead, we have to close off $\dagger$ in a recursive process with infinitely many steps.

- Start with $\dagger_0 = \dagger$.
- Given $\dagger_n$ define $\dagger_{n+1}$ as:

$$\dagger_{n+1} = \dagger_n \cup \{(a,c) \in A^2 : a \dagger_n b \dagger_n c \text{ for some } b \in A\}.$$

- Then the transitive closure of $\dagger$ is

$$\bar{\dagger} = \bigcup_{n=0}^{\infty} \dagger_n = \dagger_0 \cup \dagger_1 \cup \dagger_2 \cup \cdots$$

Consider a transitive relation $\star$ which contains $\dagger$. That is, if $a \dagger b$ then $a \star b$ and $\star$ is transitive.

- (Base case) $\star$ contains $\dagger_0$ because $\dagger_0$ is just $\dagger$, and this is true by assumption.
- (Inductive step) Assume that $\star$ contains $\dagger_n$. We want to see that if $a \dagger_{n+1} c$ then $a \star c$. By definition, $a \dagger_{n+1} c$ if either $a \dagger_n c$ or there is b so that $a \dagger_n b \dagger_n c$.
  - Case 1 ($a \dagger_n c$): Then $a \star c$ by inductive hypothesis.
  - Case 2 ($a \dagger_n b \dagger_n c$): By inductive hypothesis, $a \star b \star c$. Because $\star$ is transitive, we conclude $a \star c$.

So $\star$ contains $\bar{\dagger}$.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Some examples:

- $=$, on any set.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Some examples:

- $=$, on any set.
  This is the most important example. You should think of equivalence relations as generalizing the equality relation.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Some examples:

- $=$, on any set.
  This is the most important example. You should think of equivalence relations as generalizing the equality relation.
- The indiscrete relation on any set, where any two elements are related.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Some examples:

- $=$, on any set.
  This is the most important example. You should think of equivalence relations as generalizing the equality relation.
- The indiscrete relation on any set, where any two elements are related.
- Fix an integer $n > 1$. Then the equivalence modulo n relation, where $a \equiv b \mod n$ iff a and b have the same remainder when divided by n.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Some examples:

- $=$, on any set.
  This is the most important example. You should think of equivalence relations as generalizing the equality relation.
- The indiscrete relation on any set, where any two elements are related.
- Fix an integer $n > 1$. Then the equivalence modulo n relation, where $a \equiv b \mod n$ iff a and b have the same remainder when divided by n.
- The rearrangement relation on finite lists of integers: two lists are equivalent if they are rearrangements of each other.

Equivalence relations

Equivalence relations are a kind of relation which express that objects are the same, or should be treated equivalently.

- A binary relation $\equiv$ on A is an equivalence relation if $\equiv$ is reflexive, symmetric, and transitive.

Some examples:

- $=$, on any set.
  This is the most important example. You should think of equivalence relations as generalizing the equality relation.
- The indiscrete relation on any set, where any two elements are related.
- Fix an integer $n > 1$. Then the equivalence modulo n relation, where $a \equiv b \mod n$ iff a and b have the same remainder when divided by n.
- The rearrangement relation on finite lists of integers: two lists are equivalent if they are rearrangements of each other.
  This is the equivalence relation used in the statement of the fundamental theorem of arithmetic: when we proved that any two prime factorizations of n must be the same, what we meant is that the two lists were related in this way.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
  - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
  - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
  - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
    - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.
- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
    - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.
- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
    - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.

- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$. We've seen that $[a] \subseteq [b]$. An identical argument proves $[b] \subseteq [a]$, establishing their equality.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
  - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.
- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$. We've seen that $[a] \subseteq [b]$. An identical argument proves $[b] \subseteq [a]$, establishing their equality.
- The union of the $[a]$ is all of A

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_{\sim} = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
  - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.

- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$. We've seen that $[a] \subseteq [b]$. An identical argument proves $[b] \subseteq [a]$, establishing their equality.

- The union of the $[a]$ is all of A

  Because $a \in [a]$ for each $a \in A$.

Equivalence classes and partitions

Let $\sim$ be an equivalence relation on A. Then $\sim$ partitions A into equivalence classes.

- Consider $a \in A$. The $\sim$-equivalence class for a is:

$$[a] = [a]_\sim = \{b \in A : a \sim b\}.$$

- A partition of A is a family of nonempty pairwise disjoint subsets of A whose union is all of A.
    - Pairwise disjoint means that if we take $X \neq Y$ from the family, then $X \cap Y = \emptyset$

- Each $[a]$ is nonempty.

  Because $a \in [a]$, because $a \sim a$.
- If $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

  Fix $c \in [a] \cap [b]$. Consider $x \in [a]$. Then $a \sim x$, $a \sim c$, and $b \sim c$. By symmetry, $x \sim a$ and $c \sim b$. By transitivity, $x \sim b$, and so $b \sim x$. We've seen that $[a] \subseteq [b]$. An identical argument proves $[b] \subseteq [a]$, establishing their equality.
- The union of the $[a]$ is all of A

  Because $a \in [a]$ for each $a \in A$.

Notation: Mathematicians write $A/\sim$ for the family of $\sim$-equivalence classes.