MATH 321: HOMEWORK 3 SOLUTION

Problem 4 (Exercise 3.12 from the textbook)**.** Prove that a positive integer is square-free if and only if all the exponents in its prime factorization are 1.

Solution. I prove both directions by contrapositive.

($\Rightarrow$) Consider a positive integer n and suppose that there is a prime p in its prime factorization which has an exponent $m > 1$. Then, n is a multiple of p^m which in turn is a multiple of p^2. So n is not square-free.

($\Leftarrow$) Suppose that n is not square-free. That is, $n = a^2 b$ for some integers $a > 1$ and b. Suppose p is in the prime factorization of a, with some exponent m. And let ℓ be the largest integer so that p^ℓ divides b. (Possibly $\ell = 0$, which happens when p does not divide b.) Then, $p^{2m+\ell}$ appears in the prime factorization of n. So n has a prime in its factorization whose exponent is not 1. $\square$