MATH 1410: WORKSHEET FOR 3/31

Polynomials can be used to approximate other functions

One advantage of polynomials is that they are easy to calculate—they just involve addition, subtraction, and multiplication. Contrast this with other functions, such as trig functions and exponentials. How do you calculate what e^1 is or what $\sin(1)$ is? Of course you can leave it in exact form, but what if you want it as a decimal approximation. You could use a calculator, but that just pushes the question back a step: how does your calculator do the computation?

It turns out that polynomials can be used to approximate other functions. Since it's easy to add, subtract, or multiply—or at least, it's easy for a computer to repeatedly do these operations—you can turn a hard problem like computing $\sin(1)$ into an easy problem of doing a lot of sums/products.

Before you look at how this happens, here's a definition to make things simpler to write: For a natural number n, the *factorial* of n is

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

You can think of $n!$ as counting the number of ways to line up n many distinct objects. (There's n choices for the first object, then $n-1$ for the second, then $n-2$ for the third, ...)

(1) Use the `desmos.com` graphing calculator to graph the function e^x.

(2) Compare e^x to the function

$$f_1(x) = 1 + x.$$

Then compare to the function

$$f_2(x) = 1 + x + \frac{x^2}{2!}.$$

(Note that desmos is smart enough to know what a factorial is.) Then to the function

$$f_3(x) = 1 + x + \frac{x^2}{3!}.$$

(3) More generally, you can compare to the function

$$f_n(x) = 1 + x + \frac{x^2}{3!} + \cdots + \frac{x^n}{n!}.$$

What do you observe about the graphs as you let n be larger and larger? How large does n have to be in order for $f_n(1)$ to be within 0.001 of e^1?

(1) Graph the function $\sin x$.

(2) Compare $\sin x$ to the function

$$g_1(x) = x.$$

Then to the function

$$g_3(x) = x - \frac{x^3}{3!}.$$

Then to the function

$$g_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Then to the function

$$g_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

(3) What do you observe about the graphs of $g_n(x)$ as you let n get larger and larger? What does the general pattern for $g_n(x)$ look like? How large does n have to be in order for $g_n(1)$ to be within 0.001 of $\sin(1)$?

(1) Graph the function $\cos x$.
(2) Compare the graph of $\cos x$ to the graph of

$$h_0(x) = 1,$$

then to

$$h_2(x) = 1 - \frac{x^2}{2!},$$

then to

$$h_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$

(3) What do you observe about the graphs of $h_n(x)$ as you let n get larger and larger? What does the general pattern for $h_n(x)$ look like? How large does n have to be in order for $h_n(1)$ to be within 0.001 of $\cos(1)$?

(1) Put together what you've seen and try to find polynomials $k_n(x)$ which approximate $2e^x$.
(2) What about $\sin x + \cos x$? Can you find polynomials $\ell_n(x)$ which approximate $\sin x + \cos x$?
(3) What about $e^x + \sin x$? Can you find polynomials $m_n(x)$ which approximate $e^x + \sin x$?
(4) What about e^{2x}? Can you find polynomials $p_n(x)$ which approximate e^{2x}?

Reasonable question at this point are: How do you figure out in the first place what the polynomials to approximate e^x, $\sin x$, and $\cos x$ even are? How do you check that they really do work? Can you do this for other functions? The short answer to these questions is, this takes calculus. The long answer is, [an entire semester of calculus II].