MATH 1420: WORKSHEET FOR SECTION 5.4
THE SUBSTITUTION RULE

The substitution rule, also called change of variables, is the backward version of the chain rule.

$$\int f'(u(x)) \cdot \underbrace{u'(x)\,\mathrm{d}x}_{=\,\mathrm{d}u} = f(u(x)) + C$$

When applying it, what you need to do is figure out what $u = u(x)$ needs to be to recognize the pattern. Once you decide on u, then you need to calculate $\mathrm{d}u = u'\,\mathrm{d}x$.

Here's an example:

$$\int \frac{e^x}{\sqrt{e^x+4}}\,\mathrm{d}x.$$

$e^x + 4$ is the input to the square root, let's try that. If $u = e^x + 4$, then $\mathrm{d}u = e^x\,\mathrm{d}x$. The integral becomes

$$\int \frac{1}{\sqrt{u}}\,\mathrm{d}u = \int u^{-1/2}\,\mathrm{d}u.$$

You can find the antiderivative by using the power rule, then substitute back in for x:

$$2u^{1/2} + C = 2\sqrt{e^x+4} + C.$$

You can check your work by differentiating your answer and seeing that you get back the original integrand.

If you are calculating a definite integral using substitution, you need to make sure you use the right limits of integration. Either you need to translate back to the x-domain to use the limits for x, or you need to transform the limits for x into limits in the u-domain, by plugging in the limits for x to $u(x)$.

Let's see an example, using both methods.

$$\int_0^1 2x\sqrt{x^2+1}\,\mathrm{d}x$$

We pick $u = x^2 + 1$ and so $\mathrm{d}u = 2x\,\mathrm{d}x$. First let's solve this keeping the limits in the x-domain:

$$\begin{aligned}
\int_{x=0}^{x=1} \sqrt{u}\,\mathrm{d}u &= \left.\frac{3\sqrt{u^3}}{2}\right|_{x=0}^{x=1} \\
&= \left.\frac{3\sqrt{(x^2+1)^3}}{2}\right|_{x=0}^{x=1} \\
&= \frac{3\sqrt{8}}{2} - \frac{3\sqrt{1}}{2} \\
&= 3\sqrt{2} - \frac{3}{2}
\end{aligned}$$

If we instead want to solve this by using limits in the u-domain, we need to calculate what those limits are: $u(0) = 1$ will be the lower limit and $u(1) = 2$ will be the upper limit.

$$\begin{aligned}
\int_1^2 \sqrt{u}\,\mathrm{d}u &= \left.\frac{3\sqrt{u^3}}{2}\right|_{u=1}^{u=2} \\
&= \frac{3\sqrt{8}}{2} - \frac{3\sqrt{1}}{2} \\
&= 3\sqrt{2} - \frac{3}{2}
\end{aligned}$$

Each of the following functions came from the chain rule and is of the form $f'(u(x)) \cdot u'(x)$. Decide what $u(x)$ must be to recognize the function is in that form. Once you think you know what $u(x)$ should be, identify which part of the function is $f'(u(x))$ and which part is $u'(x)$.

(1) $3e^{6x}$
(2) $4x \sin(x^2)$
(3) $\dfrac{\cos x}{\sin x}$
(4) $\sin(\sin(x)) \cos x$

Use substitution to find each of the following indefinite integrals.

(1) $\int 3e^{6x}\,\mathrm{d}x$
(2) $\int \dfrac{\cos x}{\sin x}\,\mathrm{d}x$
(3) $\int -(x^2+2x)^5(x+1)\,\mathrm{d}x$
(4) $\int t^2 \sec^2(t^3)\,\mathrm{d}t$
(5) $\int \dfrac{2t}{(t^2+4)^3}\,\mathrm{d}t$
(6) $\int e^x + \sin x e^{\cos x}\,\mathrm{d}x$ [Hint: break the integral up along the sum.]

Find each of the following definite integrals. Make sure to use the correct limits of integration!

(1) $\int_0^{\ln 2} 2e^{2x}\,\mathrm{d}x$
(2) $\int_{-1}^{1} 5x^4(x^5-1)^3\,\mathrm{d}x$
(3) $\int_0^2 4t\sqrt{4-t^2}\,\mathrm{d}t$
(4) $\int_{\pi/2}^{\pi} \sin t e^{\cos t}\,\mathrm{d}t$