Math 1420 Exam 2

Monday, Mar 6

Name: Answer Key

This is the second midterm exam.

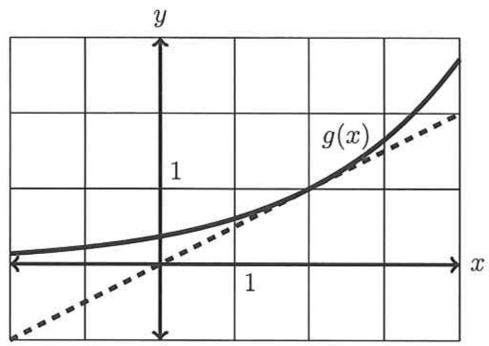
Carefully read each question and understand what is being asked before you start to solve the problem. **Show your work in an orderly fashion, and circle or mark in some way your final answers.**

No calculators nor other electronic devices are allowed.

Learning Objective	**Grade**
Conceptual Understanding	
Formal Understanding	
Rules for Calculations	
Approximations and Applications	

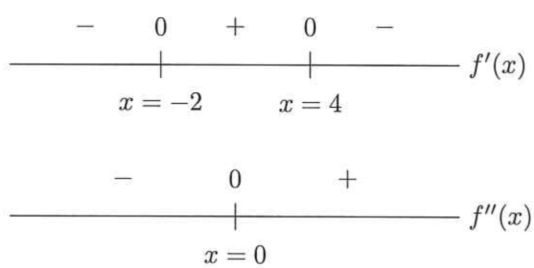
Conceptual Understanding

1. (30 points) The function $g(x)$ is given by the following graph, with the tangent line at $x = 2$ drawn in. Use this information to determine $g(2)$ and $g'(2)$.



- $g(2) =$ 1
- $g'(2) =$ 1/2

2. (40 points) Suppose you know where $f'(x)$ and $f''(x)$ are positive, negative, and zero, as given in the following sign diagrams. Use this information to say on what interval(s) $f(x)$ is increasing, decreasing, concave up, and concave down.



- Increasing: $(-2, 4)$ OR $-2 < x < 4$
- Decreasing: $(-\infty, -2) \cup (4, \infty)$ OR $x < -2$ or $4 < x$
- Concave Up: $(0, \infty)$ OR $x < 0$
- Concave Down: $(-\infty, 0)$ OR $0 < x$

3. (30 points) You are given an equation $E(x, y) = 0$ in the variables x and y which describes a curve. Do you need to first solve for y as a function of x in order to calculate the slope of the curve? Explain why or why not.

No. Implicit Differentiation let's you find the slope without doing that.

Formal Understanding

4. (10 points) Write down a function $f(x)$ which is equal to its own derivative $f'(x)$.

$F(x) = e^x$ or $F(x) = C \cdot e^x$ for any number C.

OR $F(x) = 0$:)

5. (30 points) Pretend that you don't know the power rule for differentiating x^a, where a is a nonzero constant. Use other rules for differentiating to work out the derivative of x^a.

$$y = x^a$$

$$\ln y = a \ln x$$

$$\frac{y'}{y} = a \cdot \frac{1}{x}$$

$$y' = a \cdot \frac{x^a}{x}$$

$$\underline{y' = a x^{a-1}}$$

OR

$$y = x^a = \cancel{x}(e^{\ln x})^a = e^{a \ln x}$$

$$y' = e^{a \ln x} \cdot \frac{a}{x}$$

$$= x^a \cdot \frac{a}{x}$$

$$\underline{y' = a x^{a-1}}$$

6. (30 points) Use logarithmic differentiation to differentiate $w(x) = x^{2x-1}$. You do not have to simplify fully.

$$\ln w = (2x-1)\ln x$$

$$\frac{w'}{w} = 2\ln x + \frac{2x-1}{x}$$

$$w' = x^{2x-1}\left(2\ln x + 2 - \frac{1}{x}\right)$$

7. (30 points) Use logarithmic differentiation to differentiate $v(x) = e^{x^2} \cdot \cos(2x)$. You do not have to simplify fully. **You will not get credit if you do not use logarithmic differentiation.**

$$\ln v = x^2 + \ln(\cos(2x))$$

$$\frac{v'}{v} = 2x + \frac{-2\sin(2x)}{\cos(2x)}$$

$$v' = e^{x^2} \cdot \cos(2x)\left(2x - 2\tan(2x)\right)$$

Rules for Calculations

8. (20 points) Consider $a(t) = t^5 - 3t^4 + 4t^2 + 7$. Find $a'(t)$ and $a''(t)$.

$$a'(t) = 5t^4 - 12t^3 + 8t$$

$$a''(t) = 20t^3 - 36t^2 + 8$$

9. (20 points) Consider $b(x) = 3\arccos(\sqrt{x})$. Find $b'(x)$.

$$b'(x) = \frac{-3}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{-3}{2\sqrt{x}\cdot\sqrt{1-x}}$$

$$b'(x) = \frac{-3}{2\sqrt{x-x^2}}$$

10. (20 points) Consider $c(t) = e^{2t}\sin(2t)$. Find $c'(t)$.

$$c'(t) = 2e^{2t} \cdot \sin(2t) + 2e^{2t}\cos(2t)$$

11. (20 points) Consider $d(x) = \ln(2x) - e \cdot \log_2(x)$. Find $d'(x)$ and $d''(x)$.

$$d'(x) = \frac{2}{2x} - \frac{e}{\ln(2)x} = \frac{1}{x} - \frac{e/\ln(2)}{x} = \frac{1-e/\ln(2)}{x}$$

$$d''(x) = \frac{-(1-e/\ln(2))}{x^2} = \frac{e/\ln 2 - 1}{x^2}$$

12. (20 points) Consider $f(t) = \frac{\tan t}{t}$. Find $f'(t)$.

$$f'(t) = \frac{t \cdot \sec^2 t - \tan t}{t^2}$$

Approximations and Applications

13. (40 points) The equation $2x^2 - 8xy + 3y^2 = 18$ defines a hyperbola. Find the slope of this hyperbola at both of its x-intercepts.

x-int: $y=0$: $\quad 2x^2 = 18$

$x^2 = 9$

$(3,0)$ & $(-3,0)$ $\quad x = \pm 3$

slope: $4x - 8xy' - 8y + 6y \cdot y' = 0$

$$(6y - 8x)y' = 8y - 4x$$

$$y' = \frac{8y-4x}{6y-8x}$$

$$y' = \frac{4y-2x}{3y-4x}$$

slope at $(3,0)$: $y' = \frac{-6}{-12} = \boxed{\frac{1}{2}}$

slope at $(-3,0)$: $y' = \frac{6}{12} = \boxed{\frac{1}{2}}$

30 pts

14. (60 points) (*a*) A perfectly cubic block of ice is melting, in such a way that as it melts it stays a perfect cube (albeit a smaller one). If the ice is melting at a rate of 3 cubic feet per minute, determine the rate at which the side length of the cube is decreasing when the block is 6 feet wide. Give an exact answer in feet per minute. [Hint: the volume of a cube is $V = s^3$ where s is the side length.]

Goal: $\frac{ds}{dt}$

Know: $\frac{dV}{dt} = -3 \text{ ft}^3/\text{min}$

$V = s^3$

$\frac{dV}{dt} = 3s^2 \cdot \frac{ds}{dt}$

$V = s^3$

$\frac{ds}{dt} = \frac{dV/dt}{3s^2}$

at $s = 6$ ft: $\frac{ds}{dt} = \frac{-3}{3 \cdot 6^2} = -\frac{1}{36}$ ft/min

30 pts

(*b*) For this same melting cube of ice, determine the rate at which its surface area is decreasing when the block is 6 feet wide. Give an exact answer in square feet per minute. [Hint: the surface area of a cube is $A = 6s^2$ where s is the side length.]

Goal: $\frac{dA}{dt}$

$A = 6s^2$

$\frac{A}{6} = s^2$

$s = \sqrt{\frac{A}{6}}$

$V = s^3$

$V = \left(\sqrt{\frac{A}{6}}\right)^3$

$= \frac{\sqrt{A^3}}{\sqrt{6^3}}$

$= \frac{A^{3/2}}{6\sqrt{6}}$

$\frac{dV}{dt} = \frac{\frac{3}{2} \cdot A^{1/2} \cdot \frac{dA}{dt}}{6\sqrt{6}}$

$\frac{dV}{dt} = \frac{\sqrt{A}}{4\sqrt{6}} \cdot \frac{dA}{dt}$

$\frac{dA}{dt} = \frac{4\sqrt{6} \cdot dV/dt}{\sqrt{A}}$

at $s = 6$ ft:

$A = 6 \cdot 6^2 = 216$ ft:

$\frac{dA}{dt} = \frac{4\sqrt{6} \cdot (-3)}{\sqrt{6^3}} = \frac{-12\sqrt{6}}{6\sqrt{6}} = -2 \text{ ft}^2/\text{min}$