# Math 210 Midterm 1

Name: Answer Key

This is the midterm for unit 1.

Carefully read each question and understand what is being asked before you start to solve the problem. Please show your work in an orderly fashion, and circle or mark in some way your final answers.

**No calculators nor other electronic devices are allowed.**

1. (5 points) What is the derivative of $q(x) = e^x$? What is $q'(-1)$?

$q'(x) = \boxed{e^x}$ $\qquad$ $q'(-1) = \boxed{\frac{1}{e}}$

2. (5 points) Differentiate $y(t) = t^9 - 6t^7 - t^3 + t^2 + 17t - 400$.

$y'(t) = \underline{9t^8 - 42t^6 - 3t^2 + 2t + 17}$

3. (10 points) Consider the function

$$f(x) = \frac{4\sqrt{x+2} - 8}{x-2}.$$

Compute the following limits:

$\lim_{x\to 2} f(x)$ $\quad \frac{0}{0}$

$$\lim_{x\to 2} 4\cdot\frac{(\sqrt{x+2}-2)}{x-2}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$$

$$= \lim_{x\to 2} 4\cdot\frac{\overbrace{x+2-4}^{=x-2}}{x-2(\sqrt{x+2}+2)}$$

$$= \lim_{x\to 2} \frac{4}{\sqrt{x+2}+2}$$

$$= \frac{4}{\sqrt{4}+2} = \frac{4}{4} = \boxed{1}$$

$\lim_{x\to\infty} f(x)$

$$= \lim_{x\to\infty} \frac{4\sqrt{x}}{x}$$

$$= \lim_{x\to\infty} \frac{4}{\sqrt{x}}$$

$$= \boxed{0}$$

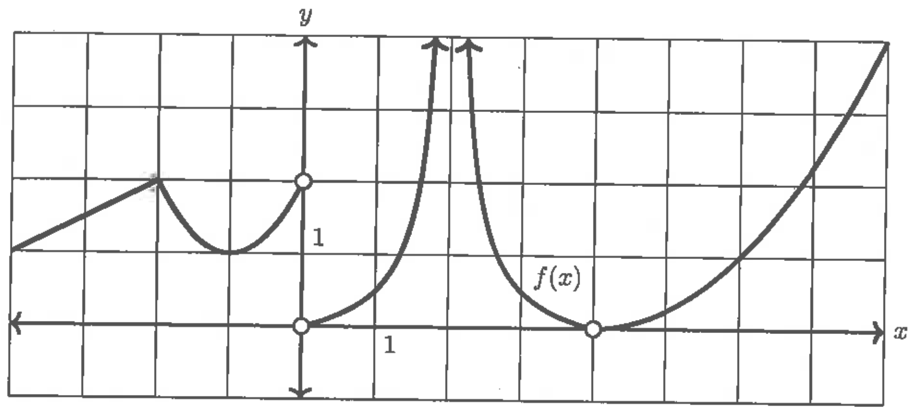


3pts conjugate

2pts algebra

4. (20 points) The function $f(x)$ with domain a subset of $[-4, 8]$ is given by the below graph.

Zench




Determine each of these limits, writing "DNE" if the limit doesn't exist:

$\lim_{x\to 0^-} f(x) =$ 2 $\qquad \lim_{x\to 0^+} f(x) =$ 0 $\qquad \lim_{x\to 0} f(x) =$ DNE

$\lim_{x\to 2} f(x) =$ $\infty$ $\qquad \lim_{x\to 4} f(x) =$ 0 $\qquad \lim_{x\to 6} f(x) =$ 1

Identify each discontinuity ~~and~~ OF $f(x)$ and classify it.

x=0 Jump

x=2 Infinite

x=4 Removable

Identify everywhere that $f(x)$ is continuous but not differentiable.

at x=-2 is a corner

#5 Note: Natural language is always somewhat ambiguous. If you had a different interpretation but your reasoning was correct for that interpretation, I went with it.

5. (10 points) You bike along a hilly route, going both uphill and downhill. Must there be a moment when your instantaneous change in elevation is 0 feet per minute? Explain why or why not.

Yes. Your change in elevation is continuous, so by the intermediate value theorem you must hit the intermediate value 0.

While you are biking a bird is flying. At the start of your bike ride the ground directly beneath the bird has a height of 2800 feet. When you finish 1 hour later the ground beneath the bird has a height of 2600 feet. Can you conclude that at some point during your trip the ground beneath the bird had a height of 2700 feet? Explain why or why not.

1 if think IVT but assume must be cont.

No, because that need not be continuous.

For example:


There is a jump discontinuity at the cliff.

6. (10 points) Consider $a(x) = \ln(x^2 + 9)$. Find $a'(x)$ and $a''(x)$.

$$a'(x) = \frac{2x}{x^2+9}$$

$$a''(x) = \frac{2(x^2+9) - 2x\cdot 2x}{(x^2+9)^2} = \frac{18-2x^2}{(9+x^2)^2}$$

7. (10 points) Consider the piecewise-defined function

$$p(x) = \begin{cases} 4x - 3 & \text{if } x \le 1 \\ 2x^2 & \text{if } x > 1 \end{cases}$$

Find all discontinuities of $p(x)$ and classify them. Write a sentence or two to explain the classification, or if there are none to explain why.

None at $x \neq 1$:

at $x = 1$:

$\lim_{x \to 1^-} p(x) = 4 \cdot 1 - 3 = 1$

$\lim_{x \to 1^+} p(x) = 2 \cdot 1^2 = 2$

Jump Discontinuity at $x = 1$

because the one-sided limits are different.

8. (5 points) Differentiate $x^e \sin x - e^{-x} \tan x$.

$\frac{d}{dx}(\ \ ) = e \cdot x^{e-1} \cdot \sin x + x^e \cos x - (-e^{-x} \cdot \tan x + e^{-x} \cdot \sec^2 x)$

$= e \cdot x^{e-1} \cdot \sin x + x^e \cdot \cos x + e^{-x} \cdot \tan x - e^{-x} \cdot \sec^2 x$

9. (5 points) Differentiate $\arctan(\sqrt{x})$.

$\frac{d}{dx}(\ \ ) = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x} + 2x\sqrt{x}}$

10. (10 points) The rules for derivatives have some redundancies. Do one of the following.

**(a)** Use the other rules for derivatives to derive the quotient rule.

**(b)** Use the other rules for derivatives to derive the rules for $b^x$ and $\log_b x$, and $\tan x$.

(a) $\frac{d}{dx}(u/v) = \frac{d}{dx}(u \cdot v^{-1}) = u' \cdot v^{-1} + u \cdot (-v^{-2} \cdot v')$

$= \frac{u'}{v} - \frac{uv'}{v^2} = \frac{u'v - uv'}{v^2}$

3pts setup
3pts product
3pts chain
2pts algebra

(b) $\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{\ln(b) \cdot x}) = \ln(b) \cdot e^{\ln(b) \cdot x} = \ln(b) \cdot b^x$ 3pts

$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln(b) \cdot x}$ 3pts

$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos x \cdot \cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$ 4pts

11. (10 points) Pick one of the following **two** functions and use the limit definition of the derivative to calculate its derivative. While it's a good way to check your work, **you will get zero points if you just use the power rule.**

$$a(x) = 3\sqrt{x} - 1 \qquad\qquad b(x) = x/2 - 2/x$$

$a(x)$: $a'(x) = \lim_{\Delta x \to 0} \frac{3\sqrt{x+\Delta x} - \cancel{1} - 3\sqrt{x} + \cancel{1}}{\Delta x} = \lim_{\Delta x \to 0} 3 \cdot \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$

$= \lim_{\Delta x \to 0} 3 \cdot \frac{\cancel{x} + \Delta x - \cancel{x}}{\Delta x \cdot (\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{3\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x+\Delta x} + \sqrt{x})}$

$= \lim_{\Delta x \to 0} \frac{3}{\sqrt{x+\Delta x} + \sqrt{x}} = \boxed{\frac{3}{2\sqrt{x}}}$

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$b(x)$: $b'(x) = \lim_{\Delta x \to 0} \frac{\cancel{\frac{x+\Delta x}{2}} - \frac{2}{x+\Delta x} - \frac{x}{2} + \cancel{\frac{2}{x}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x}{2} + \frac{2}{x} - \frac{2}{x+\Delta x}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\left(\frac{\Delta x}{2} + \frac{\cancel{2x} + 2\Delta x - \cancel{2x}}{x(x+\Delta x)}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\left(1/2 + \frac{2}{x(x+\Delta x)}\right)}{\cancel{\Delta x}}$

$= \lim_{\Delta x \to 0} \frac{1}{2} + \frac{2}{x(x+\Delta x)} = \boxed{\frac{1}{2} + \frac{2}{x^2}}$

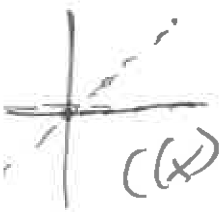
12. **Extra Credit (Up to +10)** Consider the function

$$C(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Determine the inputs where $C(x)$ is continuous, and write a short explanation for why it's continuous there.

~~[illegible]~~ only at $x=0$:   $C(0)=0$

$\lim_{x \to 0} C(x) = 0$ because the outputs for both rationals & irrationals both $\to 0 = C(0)$




At $x \neq 0$, the limits for rationals & irrationals are different: rationals $\to x$, irrationals $\to 0$, so limit does not exist

Consider next the function

$$D(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ is rational, where } p \text{ and } q \text{ are integers with no common factors} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Determine the inputs where $D(x)$ is continuous, and write a short explanation for why it's continuous there.

only if x is irrational

For irrationals, the limit is always 0.

For rationals: if x is rational, the limit is $\frac{1}{q}$

But $0 \neq \frac{1}{q}$ so the limit for all inputs DNE.

I can't draw a graph of D(x) :(

if x is irrational, the limit of rationals $\to x$ is 0, because the denominators $\to \infty$ to approach a rational value and $\frac{1}{\infty} \to 0$.



So the limit for all inputs is $0 = C(x)$.