MATH 210: 11-3 WORKSHEET
RIEMANN SUMS

A *Riemann sum* approximates the area of a region under a curve by approximating it with rectangles:

- Divide the region $[a, b]$ into N many pieces, each of length $\Delta x = \frac{b-a}{N}$.
- The endpoints of the subregions are $x_i = a + i \cdot \Delta x$, for i integers from 0 to N.
- The i-th subregion is the interval $[x_{i-1}, x_i]$.
- A recipe for the Riemann sum is:

$$\sum_{i=1}^{N} f(x_i^*) \cdot \Delta x,$$

where x_i^* is a choice of a point from the i-th region $[x_{i-1}, x_i]$.

Different choices for x_i^* give different approximations. Here's a few of the infinitely many ways to do this:

- (Left) $x_i^* = x_{i-1}$
- (Right) $x_i^* = x_i$
- (Middle) $x_i^* = \frac{x_{i-1}+x_i}{2}$
- (Max) x_i^* is the input which maximizes $f(x)$ on $[x_{i-1}, x_i]$.
- (Min) x_i^* is the input which minimizes $f(x)$ on $[x_{i-1}, x_i]$.

Once you've chosen N and how you will pick x_i^* you can fill in all your choices into the recipe to get an explicit sum you can compute.

For example, to approximate the area under $f(x) = 9 - x^2$ on $[0, 3]$ with $N = 3$ you get the following sums:

$$\text{(Left)} \qquad \sum_{i=1}^{3} f(x_{i-1}) \cdot 1 = \sum_{i=0}^{2} f(x_i) \cdot 1 = \sum_{i=0}^{2} 9 - i^2$$

$$\text{(Middle)} \qquad \sum_{i=1}^{3} f\left(\frac{x_{i-1}+x_i}{2}\right) \cdot 1 = \sum_{i=1}^{3} 9 - \left(\frac{2i-1}{2}\right)^2$$

$$\text{(Right)} \qquad \sum_{i=1}^{3} f(x_i) \cdot 1 = \sum_{i=1}^{3} 9 - i^2$$

The textbook writes L_N for the left Riemann sum for N pieces and R_N for the right Riemann sum for N pieces. The other sums aren't given names.

(1) Consider the function 2^x on the interval $[1, 5]$. Approximate the area under this function by splitting the interval into $N = 2$ pieces.
- Write down the left Riemann sum in sigma notation. Evaluate the sum.
- Write down the right Riemann sum in sigma notation. Evaluate the sum.
- Write down the middle Riemann sum in sigma notation. Evaluate the sum.
- Look at a graph of 2^x. Based on the shape of the graph, for each of the Riemann sums explain whether you think it overestimates or underestimates the true area.
- Two of the Riemann sums you wrote down are also the max Riemann sum and the min Riemann sum. Which are they?

(2) Consider the function $\sin x$ on the interval $[0, \pi]$.
- Write down the left and right Riemann sums for $N = 10$.
- Write down the left and right Riemann sums for $N = 100$.
- Write down the left and right Riemann sums for $N = 1000$.
- If you have some programming experience, write a program that computes these sums. What value does it look like they are approaching?

(3) Consider an arbitrary continuous function $f(x)$.
- Does the max Riemann sum underestimate or overestimate the true area? Or does it depend?
- Does the min Riemann sum underestimate or overestimate the true area? Or does it depend?
- Does the left Riemann sum underestimate or overestimate the true area? Or does it depend? If it depends, is there a condition on $f(x)$ that decides?
- Does the right Riemann sum underestimate or overestimate the true area? Or does it depend? If it depends, is there a condition on $f(x)$ that decides?