Math 211 Midterm 1

Name: Answer Key

This is the midterm for unit 1.

Carefully read each question and understand what is being asked before you start to solve the problem. Please show your work in an orderly fashion, and circle or mark in some way your final answers.

No calculators nor other electronic devices are allowed.

$$\int_a^b \sqrt{1+[f'(x)]^2}\,\mathrm{d}x$$

$$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,\mathrm{d}x$$

1. **(15 points)** The linear density of a certain tower made out of toy bricks is modeled by the function $\rho(y) = 12 - 18y - 12y^2 + 20y^3$ pounds per yard, where $0 \leq y \leq 1$ the vertical position in yards. Determine the mass of the tower and the height of its center of mass. Give exact answers.

$$\text{Mass} = \int_0^1 12 - 18y - 12y^2 + 20y^3 \, dy$$

$$= 12y - 9y^2 - 4y^3 + 5y^4 \Big|_0^1$$

$$= 12 - 9 - 4 + 5 = \boxed{4 \text{ pounds}}$$

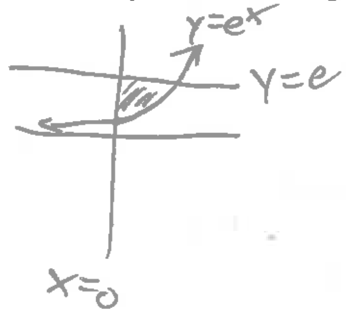
$$\text{Moment} = \int_0^1 12y - 18y^2 - 12y^3 + 20y^4 \, dy$$

$$= 6y^2 - 6y^3 - 3y^4 + 4y^5 \Big|_0^1$$

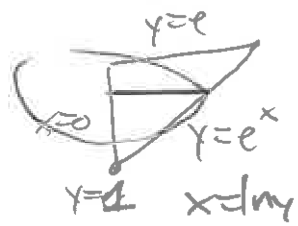
$$= 6 - 6 - 3 + 4 = 1 \text{ pound} \cdot \text{yard}$$

$$\text{C.o.M} = \frac{\text{Moment}}{\text{Mass}} = \boxed{\frac{1}{4} \text{ yards}}$$

2. (20 points) The bounded region bordered by the curves $x = 0$, $y = e$, and $y = e^x$ is rotated around the y-axis to form a solid. Set up two different formulas which give the volume of this solid, one using the method of disks and the other using the method of cylindrical shells. **Do not solve the integrals, just set them up.**

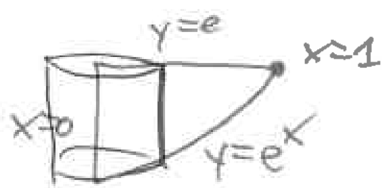


Disks:



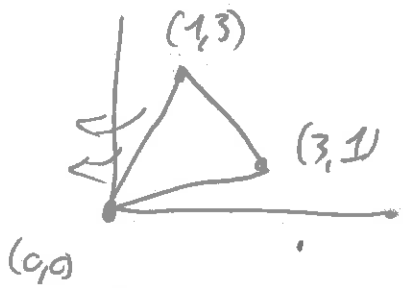
$$\text{Vol} = \int_1^e \pi \cdot (\ln y)^2 \, dy$$

Shells:

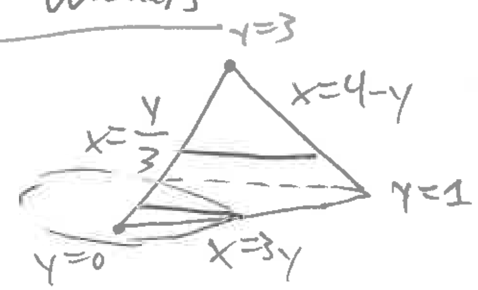


$$\text{Vol} = \int_0^1 2\pi x \cdot (e - e^x) \, dx$$

3. (15 points) The triangle with corners at $(0,0)$, $(1,3)$, and $(3,1)$ is rotated around the y-axis in order to produce a solid. **Set up a formula using integration which gives the volume of this solid. Do not solve** the integral, just **set** it up.



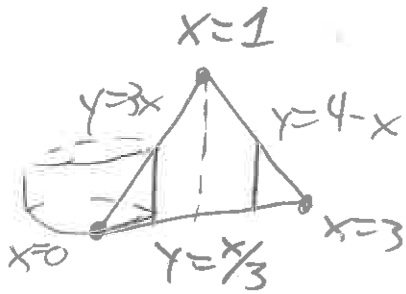
Option 1: washers



Two Pieces

$$Vol = \int_0^1 \pi\left((3y)^2 - \left(\tfrac{y}{3}\right)^2\right) dy + \int_1^3 \pi\left((4-y)^2 - (3y)^2\right) dy$$

Option 2: Shells



Two Pieces

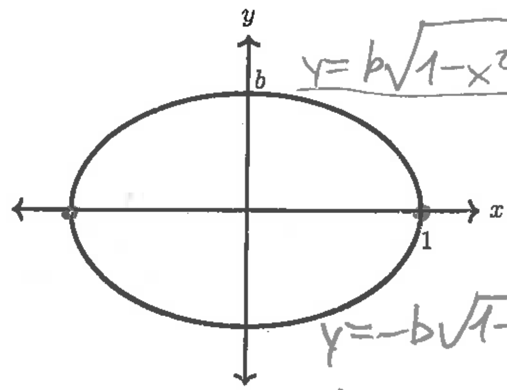
$$Vol = \int_0^1 2\pi x\left(3x - \frac{x}{3}\right) dx + \int_1^3 2\pi x\left((4-x) - \frac{x}{3}\right) dx$$

5pts divide the triangle correctly

5pts each of two integrals

4. (15 points) The equation

$$x^2 + \frac{y^2}{b^2} = 1$$



$y = b\sqrt{1-x^2}$

$y = -b\sqrt{1-x^2}$

10pts

describes an ellipse centered at the origin, as in the picture. Set up a formula using integration which gives the perimeter of the ellipse. **Do not solve the integral, just set it up.**

top & bottom are symmetric, so calculate 2× length of top.

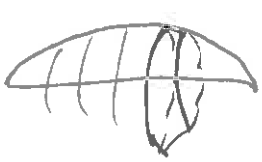
$$y' = \frac{-bx}{\sqrt{1-x^2}}$$

$$\text{Perimeter} = 2\cdot\int_{-1}^{1}\sqrt{1+\left(\frac{-bx}{\sqrt{1-x^2}}\right)^2}\,dx = 2\cdot\int_{-1}^{1}\sqrt{1+\frac{b^2x^2}{1-x^2}}\,dx$$

5pts

Rotating the ellipse around either one of the axes produces a surface known as ellipsoid. Set up a formula using integration which gives the surface area of the ellipsoid. **Do not solve the intgral, just set it up.**

Rotate top around x-axis.



$$S.A. = \int_{-1}^{1} 2\pi\cdot b\sqrt{1-x^2}\cdot\sqrt{1+\frac{b^2x^2}{1-x^2}}\,dx$$

Same y, y'.

$$= \int_{-1}^{1} 2\pi b\cdot\sqrt{1-x^2+b^2x^2}\,dx = 2\pi b\int_{-1}^{1}\sqrt{1-(1-b^2)x^2}\,dx$$

5. (15 points) You are taking care of bees on the Simon's Rock Community Farm. You just set up a queen in a new hive, and while the hive is youngexponential growth is a good model for the population. On day 0, when you started the new hive, you counted 100 bees in the hive. One week later you counted 120 bees. Write a formula which describes the population $B(t)$ of bees t days after you started the hive. Use this formula to predict the population two weeks after the start. Give an exact answer.

$B(0) = 100 \quad B(7) = 120$

$B(t) = 100 \exp(kt)$

$B(t) = 100 \cdot \exp(\ln(\frac{6}{5})/7) \cdot t)$

$120 = 100 \cdot \exp(7k)$

$7k = \ln\left(\frac{120}{100}\right) = \ln\left(\frac{6}{5}\right)$

$k = \ln\left(\frac{6}{5}\right)/7$

$B(14) = 100 \cdot \exp(\ln(\frac{6}{5}) \cdot \overbrace{14/7}^{=2}) = 100 \cdot (\exp(\ln(\frac{6}{5})))^2$

$= 100 \cdot \left(\frac{6}{5}\right)^2 = 100 \cdot \frac{36}{25} = 4 \cdot 36 = 144$ bees

6. (20 points) An oracle tells you that the only solution to $x^3 + 4x - 16 = 0$ is $x = 2$. Determine the area in the bounded region bordered by the curves

$$y = \sqrt{x} \qquad \text{and} \qquad y = \frac{x\sqrt{x^2+4}}{4}.$$

Give an exact answer and simplify fully. [Hint: your final answer should be a rational number.]

$$\sqrt{x} = \frac{x\sqrt{x^2+4}}{4}$$

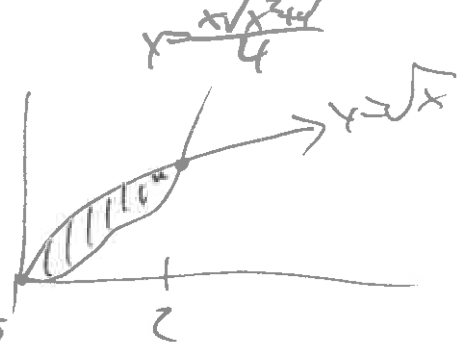
$$4\sqrt{x} = x\sqrt{x^2+4}$$

$$16x = x^2(x^2+4)$$

$$x^4 + 4x^2 - 16x = 0$$

$$x(x^3+4x-16) = 0$$

$$x = 0 \text{ or } x = 2$$



$$\text{Area} = \int_0^2 \sqrt{x} - \frac{x\sqrt{x^2+4}}{4}\, dx.$$

Split into two integrals

$$\int_0^2 \sqrt{x}\, dx = \frac{2}{3} x^{3/2} \Big|_0^2$$

$$= \frac{2}{3} \cdot 2^{3/2}$$

$$= \frac{2}{3} \cdot 2\sqrt{2}$$

$$= \frac{4\sqrt{2}}{3}$$

$$\int_0^2 \frac{x\sqrt{x^2+4}}{4}\, dx = \int_4^8 \frac{\sqrt{u}}{8}\, du = \frac{2}{3} \cdot \frac{1}{8} \cdot u^{3/2} \Big|_4^8$$

$u = x^2+4$

$du = 2x\,dx$

$$= \frac{1}{12} \cdot 8^{3/2} - \frac{1}{12} \cdot 4^{3/2} = \frac{1}{12} \cdot (2\sqrt{2})^3 - \frac{1}{12} \cdot 2^3$$

$$= \frac{8 \cdot 2\sqrt{2}}{12} - \frac{8}{12} = \frac{4\sqrt{2}}{3} - \frac{2}{3}$$

$$\text{Area} = \frac{4\sqrt{2}}{3} - \left(\frac{4\sqrt{2}}{3} - \frac{2}{3}\right) = \boxed{\frac{2}{3}}$$