MATH 211: 11-8 AND 11-10 WORKSHEET

If a function $f(x)$ has a power series (centered at c) with positive radius of convergence, that power series is the Taylor series (centered at c) for $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Here are a few important Taylor series. More are on page 585 of the textbook. Unless otherwise stated, the radius of convergence is ∞.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \pm \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{(RoC = 1)}$$

$$\ln(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \qquad \text{(RoC = 1)}$$

(1) In lecture we checked the Taylor series at $x = 0$ for e^x. Check that the Taylor series on the other page for $\sin x$ and $\cos x$ are also correct.
(2) Knowing the Taylor series at $x = 0$ for e^x, get the series for e^{3x} and e^{x^2}. [Hint: substitute $3x$ or x^2 into the Taylor series.]
(3) Use a computer tool to graph the first several *Taylor polynomials* (= the polynomials obtained by looking at a partial sum of a Taylor series) for e^x.
(4) Use a computer tool to graph the first several Taylor polynomials for $\sin x$. Explain how you could use a Taylor polynomial for $\sin x$ to approximately calculate $\sin(\pi/6)$.
(5) Use a computer tool to graph the first several Taylor polynomials for $\cos x$. Explain how you could use a Taylor polynomial for $\sin x$ to approximately calculate $\cos(8\pi - \pi/6)$.

These problems give examples of applications of Taylor series, to try to convince you that they are a powerful tool.

(1) (Differential equations) Show that the function $y(x)$ with $y(0) = 1$ which makes the equation $y' = y$ true is $y(x) = e^x$.
  (a) Considering a series for $y(x)$:
$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$
  (b) Differentiate it term-by-term to get a series for $y'(x)$.
  (c) Using that the two series are equal (because $y' = y$), figure out the coefficients a_n. [Hint: start by figuring out a_0, then use it to get a_1, then use that to get a_2, and so on. You should notice a general pattern.]
(2) (Integration) e^{-x^2} is probably the most famous function which doesn't have an easily written antiderivative. But you can use Taylor series to approximate, say, $\int_0^1 e^{-x^2}\,dx$.
  (a) Use the Taylor series for e^x to get a series representing e^{-x^2}.
  (b) Antidifferentiate that series term-by-term to get a series representing $\int e^{-x^2}\,dx$. Explicitly compute the coefficients a_n out to $n = 5$.
  (c) Use the degree 5 polynomial approximation to that series to approximate $\int_0^1 e^{-x^2}\,dx$. Check your answer is accurate by comparing to what a computer tool gives.[1]

[1] A topic we skipped over due to time is *error estimation*. Part of the power of Taylor series is that they give methods for estimating how accurate an approximation a Taylor polynomial is. In a real application, you'd want to use that error estimate to check how accurate your answer is, rather than just compare to a computer tool.