MATH 210: 10-30 WORKSHEET
4.3 RULES FOR INTEGRATION

A *definite integral* is a measure of area under a curve. It is computed by finding the values of an antiderivative at the endpoints:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

An *indefinite integral* is another name for the general antiderivative:

$$\int f(x)\,\mathrm{d}x = F(x) + C,$$

where C stands for an arbitrary constant.

To compute integrals, the game is thus to compute antiderivatives. Our first tool for this is to run the rules for derivatives backward.

Atomic Rules

$$\int x^\alpha\,\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad (\alpha \neq -1)$$

$$\int x^{-1}\,\mathrm{d}x = \ln|x| + C$$

$$\int e^x\,\mathrm{d}x = e^x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

Combination Rules

$$\int c\cdot f(x)\,\mathrm{d}x = c\cdot\int f(x)\,\mathrm{d}x$$

$$\int \big(f(x)+g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

$$\int \big(f(x)-g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x$$

The other atomic rules for derivatives come up less often in integration, but they are still good to notice. For example, you can know that

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C$$

by remembering $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$.

The other combination rules are more difficult. We will study them separately. The backward version of the chain rule is the topic of the next section, and the backwards version of the product rule is covered in Calc 2.

Compute the indefinite integrals of the following functions.

- $a(x) = 2x + x^2$
- $b(x) = \frac{3}{\sqrt{x}}$
- $c(x) = 4e^x - \frac{1}{x}$
- $d(x) = \sin x + \cos x$
- $f(x) = \pi \sec^2 x - \cot x \csc x$
- $g(x) = 17$
- $h(x) = (x-2)^2$

The rules on the front page can be checked by differentiation. By the fundamental theorem of calculus, differentiation and integration are opposite procedures. So if you differentiate an indefinite integral you should get back the original function.

(1) Suppose $F(x)$ is an antiderivative of $f(x)$, $G(x)$ is an antiderivative of $g(x)$, and c is a constant. Confirm that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and $cF(x)$ is an antiderivative of $cf(x)$. Explain why this gives the constant multiple and sum rules for integrals.
(2) Determine an antiderivative of $(\alpha + 1)x^\alpha$ (where $\alpha \neq -1$) by thinking about what you would differentiate to get that. Explain why this gives you the power rule for integrals.
(3) Explain why the integration rules for e^x, $\cos x$, and $\sin x$ all work.

Finally, I want to leave you with a warning: computing antiderivatives is generally harder than computing derivatives. Very often there isn't an atomic rule you can run backward!

(1) Try to determine an antiderivative of $\ln x$ by looking at the atomic rules for derivatives.
(2) After you give up at the previous problem, compute the derivative of $x \ln x - x$. Use your answer to this to determine the general antiderivative of $\ln x$.
[If this looks totally out of nowhere: in Calc 2 you will learn a method for finding antiderivatives (namely, *integration by parts*) that could be used to determine the general antiderivative of $\ln x$.]