MATH 210: 11-4 WORKSHEET
4.4 INTEGRATION BY CHANGE OF VARIABLES

Last Friday we saw the method of *change of variables*, also called *substitution*.
To integrate

$$\int f(u(x))\,\underbrace{u'(x)\,\mathrm{d}x}_{=\,\mathrm{d}u} = \int f(u)\,\mathrm{d}u,$$

you use the substitution

$$u = u(x) \qquad\qquad \mathrm{d}u = u'(x)\,\mathrm{d}x.$$

to change from the x domain to the u domain. Then, you need to find an antiderivative $F(u)$ for $f(u)$, and then you substitute back to the x domain to get the antiderivative $F(u(x))$.

When using change of variables to calculate definite integrals, you also need to transfer the limits of integration from the x domain to the u domain:

$$\int_a^b f(u(x))\,\underbrace{u'(x)\,\mathrm{d}x}_{=\,\mathrm{d}u} = \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u = F(u)\Big|_{u(a)}^{u(b)}.$$

Note that when you do this, you should not translate the antiderivative back to the x domain.

First let's check with a simple example that the change of limits is necessary.

(1) Calculate $\int_0^2 2x(x^2+1)\,\mathrm{d}x$ by distributing the multiplication then using the power rule.
(2) Calculate $\int_0^2 2x(x^2+1)\,\mathrm{d}x$ by using the substitution $u = x^2+1$ and $\mathrm{d}u = 2x\,\mathrm{d}x$. Make sure you correctly translate the limits to the u domain!
(3) Compare to what you would've gotten if you hadn't changed the limits: calculate $\int_0^2 u\,\mathrm{d}u$.

Calculate the following definite integrals by change of variables.

(1) $\int_1^e \sqrt{\ln x}\cdot\frac{1}{x}\,\mathrm{d}x$
(2) $\int_0^1 xe^{2x^2-1}\,\mathrm{d}x$
(3) $\int_1^{\ln 3} e^{3x}\,\mathrm{d}x$
(4) $\int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin x}\,\mathrm{d}x$
(5) $\int_0^{\pi/4} \tan x\,\mathrm{d}x$

As an alternative method, you can keep the limits of integration in the x domain, but then you have to translate the antiderivative to the x domain before evaluating at the endpoints.

$$\int_a^b f(u(x))\,\underbrace{u'(x)\,\mathrm{d}x}_{=\mathrm{d}u} = \int_{x=a}^{x=b} f(u)\,\mathrm{d}u = F(u)\Big|_{x=a}^{x=b} = F(u(x))\Big|_a^b.$$

(1) Calculate $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$ by translating to the u-domain with new limits in the u-domain.
(2) Calculate $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$ by keeping the limits of integration in the x-domain. Which method do you prefer?