Math 210 Midterm 1

Friday 27 September

Name: Answer Key

This is the midterm for unit 1.

Carefully read each question and understand what is being asked before you start to solve the problem. Please show your work in an orderly fashion, and circle or mark in some way your final answers.

No calculators nor other electronic devices are allowed.

1. (10 points) (*a*) What is the derivative of $f(t) = e^t$?

$f'(t) = e^t$

(*b*) What is the derivative of

$$g(x) = x^{2024} - 4x^7 + 6x^3 + 20x - 400?$$

$g'(x) = 2024x^{2023} - 28x^6 + 18x^2 + 20$

2. (10 points) What is the derivative of $k(t) = \sqrt{t-2} \cdot \arctan(t) + \tan(3t)$?

$k'(t) = \frac{\sqrt{t-2}}{1+t^2} + \frac{\arctan(t)}{2\sqrt{t-2}} + 3\sec^2(3t)$

3. (10 points) (*a*) Consider the following expression, where H is positive and infinite. Determine whether the expression is infinitesimal, finite but non-infinitesimal, or infinite. **Show your work.**

$$\sqrt{H+100}-\sqrt{H}$$

$\infty - \infty$ indeterminate form

$$= \sqrt{H+100}-\sqrt{H} \cdot \left(\frac{\sqrt{H+100}+\sqrt{H}}{\sqrt{H+100}+\sqrt{H}}\right) = \frac{H+100-H}{\sqrt{H+100}+\sqrt{H}} = \frac{100}{\sqrt{H+100}-\sqrt{H}}$$

$\frac{\text{Finite}}{\text{infinite}}$ is infinitesimal

(*b*) Find the following standard part, where ε is a nonzero infinitesimal. **Show your work.**

$$\mathrm{st}\left(\frac{\varepsilon^4-4\varepsilon}{\varepsilon^2+2\varepsilon}\right)$$

$\frac{0}{0}$ indeterminate form

$$= \mathrm{st}\left(\frac{\varepsilon^3-4}{\varepsilon+2}\right) = \frac{0-4}{0+2} = -2$$

4. (10 points) Compute the derivative of $h(x) = x^{\ln x}$.

$$\ln h = \ln(x^{\ln x})$$
$$= (\ln x)^2$$

$$\frac{h'}{h} = \frac{2\ln x}{x}$$

$$h' = \frac{2\ln x}{x} \cdot x^{\ln x}$$

$$= 2\ln x \cdot x^{\ln x - 1}$$

5. (10 points) Give an equation for the the line **tangent** to the curve $y = 4\cos(\frac{\pi}{2}x + \frac{\pi}{6}) - 2\sqrt{3}$ at $x = 0$.
[Hint: $\sin(\pi/6) = 1/2$ *and* $\cos(\pi/6) = \sqrt{3}/2$.*]*

$y' = -4 \cdot \frac{\pi}{2} \sin(\frac{\pi}{2}x + \frac{\pi}{6})$

$= -2\pi \sin(\frac{\pi}{2}x + \frac{\pi}{6})$

$y(0) = 4\cos(\frac{\pi}{6}) - 2\sqrt{3}$

$= 4 \cdot \frac{\sqrt{3}}{2} - 2\sqrt{3}$

$= 2\sqrt{3} - 2\sqrt{3} = \underline{0}$

$y'(0) = -2\pi \sin(0 + \frac{\pi}{6})$

$= -2\pi(1/2)$

$= \underline{-\pi}$

$y = y'(0)(x - 0) + y(0)$

$\boxed{y = -\pi x}$

6. (10 points) Find the slope of the curve $\frac{x^2 y}{2} + x^3 - y = 0$ at the point $(1, 2)$.

implicit differentiation

$\frac{x^2 y'}{2} + xy + 3x^2 - y' = 0$

Plug in $x = 1$, $y = 2$:

$\frac{y'}{2} + 2 + 3 - y' = 0$

$(\frac{1}{2} - 1)y' = -5$

$-\frac{y'}{2} = -5$

$\boxed{y' = 10}$

Alt: solve for y' first

$(\frac{x^2}{2} - 1)y' = -3x^2 - xy$

$y' = \frac{-3x^2 - xy}{x^2/2 - 1}$

@ $(1, 2)$:

$y' = \frac{-3 - 2}{-1 + 1/2} = \frac{-5}{-1/2} = \underline{10}$

7. (10 points) Compute the first and second derivatives of

$$\ell(u) = u^2 \ln u.$$

$\ell'(u) = \frac{u^2}{u} + 2u\ln u = \underline{u + 2u\ln u}$

$\ell''(u) = 1 + \frac{2u}{u} + 2\ln u = \underline{3 + 2\ln u}$

8. (10 points) Compute the derivative of

$$j(x) = \frac{\pi^{ex}}{\csc x}. \quad = \pi^{ex} \cdot \sin x$$

$j'(x) = e\ln(\pi) \cdot \pi^{ex} \cdot \sin x + \pi^{ex} \cdot \cos x$

$= \underline{\pi^{ex}(e\ln(\pi)\sin x + \cos x)}$

Alt: $j'(x) = \dfrac{\cancel{e}\ln(\pi) \cdot \pi^{ex}\csc x + \pi^{ex}\csc x \cot x}{\csc^2 x}$

Quotient Rule

$= \dfrac{e\ln(\pi) \cdot \pi^{ex}}{\csc x} + \dfrac{\pi^{ex}\cot x}{\csc x} = e\ln(\pi)\pi^{ex} \sin x + \pi^{ex} \cdot \dfrac{\cos x/\sin x}{1/\sin x}$

an also

o log. diff.

$= \underline{e\ln(\pi)\pi^{ex} \cdot \sin x + \pi^{ex} \cdot \cos x}$

9. (10 points) **Do exactly one of the following. If you attempt both, cross out the one you don't wish me to grade.**

(*a*) Use the other rules for derivatives to derive the differentiation rule for $\arcsin x$.

(*b*) Use the other rules for derivatives to derive the quotient rule.

(a) Inverse Function Rule $\quad \frac{d}{dx}\sin x = \cos x$

Pyth. Id.

$$\text{So } \frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

(b) $y = \frac{u}{v} \rightarrow y' = \frac{u'v - uv'}{v^2}$ Alt: product rule + chain rule

log. diff.

$\ln y = \ln u - \ln v$

$$\frac{y'}{y} = \frac{u'}{u} - \frac{v'}{v}$$

$$y' = \frac{u}{v}\left(\frac{u'}{u} - \frac{v'}{v}\right)$$

$$= \frac{u'}{v} - \frac{uv'}{v^2}$$

$$= \frac{u'v - uv'}{v^2}$$

$y = u \cdot v^{-1}$

$y' = u' \cdot v^{-1} + u \cdot (-v^{-2})v'$

$$= \frac{u'}{v} - \frac{uv'}{v^2}$$

$$= \frac{u'v - uv'}{v^2}$$

10. (10 points) Use the definition of the derivative in terms of a standard part to calculate the derivative of one of the following functions. If you attempt both, clerly mark which one you want me to grade. While it's a good way to check your work, **you will get zero points if you just use the power rule.**

$$b(x) = x - \frac{1}{x} \qquad\qquad c(x) = \frac{3}{\sqrt{5x}}$$

$$b'(x) = st\left(\frac{x+\Delta x - \frac{1}{x+\Delta x} - x + \frac{1}{x}}{\Delta x}\right) = st\left(\frac{\Delta x + \frac{-x+x+\Delta x}{x(x+\Delta x)}}{\Delta x}\right)$$

$$= st\left(\frac{\Delta x + \frac{\Delta x}{x(x+\Delta x)}}{\Delta x}\right) = st\left(1 + \frac{1}{x(x+\Delta x)}\right) = 1 + \frac{1}{x^2}$$

$$c'(x) = st\left(\frac{\frac{3}{\sqrt{5(x+\Delta x)}} - \frac{3}{\sqrt{5x}}}{\Delta x}\right) = st\left(\frac{3\sqrt{5x} - 3\sqrt{5(x+\Delta x)}}{\Delta x\sqrt{5x(5)(x+\Delta x)}}\right)$$

$$= st\left(\frac{3}{5}\cdot\frac{\sqrt{5x}-\sqrt{5(x+\Delta x)}}{\Delta x\sqrt{(x+\Delta x)x}}\cdot\frac{\sqrt{5x}+\sqrt{5(x+\Delta x)}}{\sqrt{5x}+\sqrt{5(x+\Delta x)}}\right) = \frac{3}{5}\, st\left(\frac{5x-5x-5\Delta x}{\Delta x\sqrt{(x+\Delta x)x}\left(\sqrt{5x}+\sqrt{5(x+\Delta x)}\right)}\right)$$

$$= \frac{3}{5}\, st\left(\frac{-5}{\sqrt{(x+\Delta x)x}\left(\sqrt{5x}+\sqrt{5(x+\Delta x)}\right)}\right) = \frac{3}{5}\cdot\frac{-5}{\sqrt{x^2}\cdot 2\sqrt{5x}}$$

$$= \boxed{-\frac{3}{2\sqrt{5x^3}}}$$

(Extra space. Clearly **label** which problem the work is for.)