# MATH 210
# RULES FOR LIMITS

## Limit definition

The *limit* of $f(x)$ as $x$ approaches $c$ is equal to $L$,

$$\lim_{x \to c} f(x) = L,$$

if whenever $x \neq c$ is infinitely close to $c$ you have $f(x)$ is infinitely close to $L$. If this does not happen for a unique $L$, then we say the limit is *undefined* or *does not exist*.

## Computing limits

To compute $\lim_{x \to c} f(x)$:

(1) Let $x \neq c$ be infinitely close to $c$.
(2) Simplify $f(x)$, if necessary.
(3) Compute the standard part $\text{st}(f(x))$.

## One-sided limits

- *Left limit*: $\lim_{x \to c^-} f(x) = L$ if whenever $x < c$ is infinitely close to $c$ you have $f(x) \approx L$.
- *Right limit*: $\lim_{x \to c^+} f(x) = L$ if whenever $x > c$ is infinitely close to $c$ you have $f(x) \approx L$.
- **Fact:** $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$.

## Rules for limits

Because limits are certain standard parts, the rules for standard parts immediately give rules for limits. Here, assume that $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ are both defined. The analogous rules hold for one-sided limits.

- **Constant multiple rule.** Let $k$ be a real number.

$$\lim_{x \to c} k \cdot f(x) = k \cdot \lim_{x \to c} f(x)$$

- **Sum rule.**

$$\lim_{x \to c} f(x) + g(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

- **Difference rule.**

$$\lim_{x \to c} f(x) - g(x) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$$

- **Multiplication rule.**

$$\lim_{x \to c} f(x) \cdot g(x) = \left(\lim_{x \to c} f(x)\right) \cdot \left(\lim_{x \to c} g(x)\right)$$

- **Division rule.** Assume $\lim_{x \to c} g(x) \neq 0$.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

- **Power rule.** Let $k$ be a real number

$$\lim_{x \to c} f(x)^k = \left(\lim_{x \to c} f(x)\right)^k$$

- **Root rule.** Let $n$ be a positive integer.

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$$

## Practice problems

Compute the following limits, or say why the limit does not exist.

(1) $\lim_{x \to 3} x^3 - 3x$

(2) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

(3) $\lim_{x \to 0} \frac{3 + 2/x}{1 - 3/x}$

(4) $\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}$

(5) $\lim_{x \to 0} \frac{|x|}{x}$

(6) $\lim_{x \to 0^+} \frac{|x|}{x}$

(7) $\lim_{x \to -3} 5\sqrt{x + 3}$

(8) $\lim_{x \to -3^+} 5\sqrt{x + 3}$

(9) $\lim_{x \to a} \sqrt{|a - x|}$

Compute the following limits. [Hint: If you recognize that they are instances of the definition of the derivative, you can use rules for differentiating!]

(1) $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$

(2) $\lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$

(3) $\lim_{\Delta x \to 0} \frac{3\sqrt{x + \Delta x + 1} - 3\sqrt{x + 1}}{\Delta x}$

(4) $\lim_{\Delta x \to 0} \frac{e^{x + \Delta x} - e^x}{\Delta x}$