MATH 210
RULES FOR STANDARD PARTS

Properties of $\approx$

Two numbers a and b are *infinitely close*, $a \approx b$, if their difference $a - b$ is infinitesimal.

(1) **Basic properties.**
   (a) a is infinitesimal if and only if $a \approx 0$;
   (b) If a and b are real and $a \approx b$ then $a = b$.

(2) **Equality-like properties.**
   (a) $a \approx a$;
   (b) If $a \approx b$ then $b \approx a$;
   (c) If $a \approx b$ and $b \approx c$ then $a \approx c$.

(3) **Size properties.** Assume $a \approx b$.
   (a) If a is infinitesimal then so is b;
   (b) If a is finite then so is b;
   (c) If a is infinite then so is b.

Standard parts

Standard part principle. Every finite hyperreal number a is infinitely close to exactly one real number. We call this number the *standard part* of a, and denote it $\mathrm{st}(a)$.

(1) **Basic properties.** Let a be finite.
  (a) $\mathrm{st}(a)$ is a real number;
  (b) $a = \mathrm{st}(a) + \varepsilon$ for some infinitesimal ε;
  (c) If a is real then $a = \mathrm{st}(a)$.

(2) **Arithmetic properties.** Let a and b be finite.
  (a) $\mathrm{st}(-a) = -\mathrm{st}(a)$;
  (b) $\mathrm{st}(a+b) = \mathrm{st}(a) + \mathrm{st}(b)$;
  (c) $\mathrm{st}(a-b) = \mathrm{st}(a) - \mathrm{st}(b)$;
  (d) $\mathrm{st}(a \cdot b) = \mathrm{st}(a) \cdot \mathrm{st}(b)$;
  (e) $\mathrm{st}(a/b) = \mathrm{st}(a)/\mathrm{st}(b)$, provided $\mathrm{st}(b) \neq 0$;
  (f) $\mathrm{st}(a^n) = \mathrm{st}(a)^n$;
  (g) $\mathrm{st}(\sqrt[n]{a}) = \sqrt[n]{\mathrm{st}(a)}$, provided $a \geq 0$;
  (h) If $a \leq b$ then $\mathrm{st}(a) \leq \mathrm{st}(b)$.