MATH 210
WEEK 2 WRITING ASSIGNMENT SOLUTION

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Problem. *Consider the step function $u(x)$ defined as*

$$u(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Determine where $u(x)$ is differentiable and justify yoour answer.

Solution. When $x < 0$ so is $x + \Delta x$ and so both inputs have an output of 0. In this case, $u(x+\Delta x) - u(x) = 0$ and so $u'(x) = 0$. Similarly when $x > 0$ we get that $u(x)$ and $u(x+\Delta x)$ are both 1, and so $u'(x) = 0$. In all, this shows that $u(x)$ is differentiable when $x \neq 0$.

At $x = 0$ two things go wrong. First, some values of Δx, specifically when $\Delta x > 0$, make $\frac{\Delta y}{\Delta x}$ infinite, and so its standard part is undefined. Second, when $\Delta x < 0$ we get that $\operatorname{st}(\frac{\Delta y}{\Delta x}) = 0$, so the value of the standard part depends on the choice of Δx. Thus $u(x)$ is not differentiable at $x = 0$. Let's now see the calculations which justify those two statements: In both cases, $u(0) = 0$.

- If $\Delta x > 0$ then $u(0 + \Delta x) = 1$. So $\Delta y = 1 - 0 = 1$ and so $\frac{\Delta y}{\Delta x}$ is a finite value divided by an infinitesimal, so it is infinite.
- If $\Delta x < 0$ then $u(0 + \Delta x) = 0$. So $\Delta y = 0 - 0 = 0$ and so $\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$, whence its standard part is also 0. □

Problem (Extra Credit)**.** *Find the derivative of $f(x) = |x^2 - 1|$, identifying where $f(x)$ is not differentiable.*

Solution. A polynomial inequality calculation readily shows that $x^2-1 > 0$ when $-1 < x < 1$ and $x^2 - 1 < 0$ when $|x| > 1$. We will handle those two cases separately:

- $(-1 < x < 1)$ In this case $f(x) = x^2 - 1$. We can compute the derivative with a quick calculation:

$$f'(x) = \operatorname{st}\left(\frac{(x + \Delta x)^2 - 1 - (x^2 - 1)}{\Delta x}\right) = \operatorname{st}\left(\frac{2x\Delta x + \Delta x^2}{\Delta x}\right) = \operatorname{st}(2x + \Delta x) = 2x.$$

- $(|x| > 1)$ In this case $f(x) = -x^2 + 1$. We can compute the derivative with a quick calculation:

$$f'(x) = \operatorname{st}\left(\frac{-(x + \Delta x)^2 + 1 - (-x^2 + 1)}{\Delta x}\right) = \operatorname{st}\left(\frac{-2x\Delta x - \Delta x^2}{\Delta x}\right) = \operatorname{st}(-2x - \Delta x) = -2x.$$

Last we come to the case $x = \pm 1$. Depending on whether Δx is positive or negative we will use one of the two formulas from the previous case for calculating the standard part. (Specifically, at $x = -1$ when $\Delta x < 0$ we have $-1 + \Delta x < -1$ and so we use the formula $f(x) = -x^2 + 1$ for the calculation and when $\Delta x > 0$ we have $-1 < -1 + \Delta x < 1$ and so we use the formula $f(x) = x^2 - 1$. At $x = 1$ the two are swapped.) Thus we get that depending on whether Δx is positive or negative we will have either $2x$ or $-2x$ as the standard part.

Now we just check those give different values: if $x = -1$ then $2x = -2 \neq 2 = -2x$ and if $x = 1$ then $2x = 2 \neq -2 = -2x$. Thus we see that $f(x)$ is not differentiable at $x = \pm 1$.

In all, we calculated that:

$$f'(x) = \begin{cases} 2x & \text{if } |x| < 1 \\ -2x & \text{if } |x| > 1 \end{cases}$$

and $f'(x)$ is undefined at $x = \pm 1$. $\square$