MATH 210
WEEK 3 WRITING ASSIGNMENT SOLUTION

Problem. *Use the definition of the derivative to explain why the sum rule* $\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)+g(x)\right) = f'(x) + g'(x)$ *is valid.*

Solution. The derivative of $f(x) + g(x)$ is:

$$\begin{aligned}
\mathrm{st}\left(\frac{(f(x+\Delta x)+g(x+\Delta x)) - (f(x)+g(x))}{\Delta x}\right) &= \mathrm{st}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x} + \frac{g(x+\Delta x)-g(x)}{\Delta x}\right) \\
&= \mathrm{st}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right) + \mathrm{st}\left(\frac{g(x+\Delta x)-g(x)}{\Delta x}\right) \\
&= f'(x) + g'(x).
\end{aligned}$$

The first equality is just algebra, the second is the rule that the standard part of a sum is the sum of the standard parts, and the last is the definition of the derivative. □

Problem (Extra credit)**.** *Use the definition of the derivative to explain why the power rule* $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$*, where* n *is a positive integer, is valid.*

Solution 1, using the binomial theorem. To calculate the derivative of x^n we use the binomial theorem. It states, in the form appropriate to this problem, that

$$(x+\Delta x)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}\Delta x + \cdots + \binom{n}{k}x^{n-k}\Delta x^k + \cdots + \binom{n}{n}\Delta x^n.$$

Here, $\binom{n}{k}$ is the k-th binomial coefficient for n. All that matters for our calculation is that $\binom{n}{0} = 1$ and $\binom{n}{1} = n$.

We use the binomial theorem to calculate $\mathrm{st}(\frac{(x+\Delta x)^n - x^n}{\Delta x})$. Note that the x^n terms completely cancel out, since we have $x^n - x^n$. After this all terms in the numerator have a Δx, so we can cancel out a common Δx from the numerator and the denominator. After doing so, the only term without a Δx is the x^{n-1} term, namely nx^{n-1}. Thus, it is the only term that remains nonzero when taking the standard part, so it is the derivative.

Here's the same explanation in symbols with lots of dotdotdots, in case that's clearer:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} x^n &= \mathrm{st}\left(\frac{(x+\Delta x)^n - x^n}{\Delta x}\right) \\
&= \mathrm{st}\left(\frac{\binom{n}{0}x^n + \binom{n}{1}x^{n-1}\Delta x + \cdots + \binom{n}{k}x^{n-k}\Delta x^k + \cdots + \binom{n}{n}\Delta x^n - x^n}{\Delta x}\right) \\
&= \mathrm{st}\left(\frac{nx^{n-1}\Delta x + \cdots + \binom{n}{k}x^{n-k}\Delta x^k + \cdots + \binom{n}{n}\Delta x^n}{\Delta x}\right) \\
&= \mathrm{st}\left(nx^{n-1} + \cdots + \binom{n}{k}x^{n-k}\Delta x^{k-1} + \cdots + \binom{n}{n}\Delta x^{n-1}\right) \\
&= nx^{n-1} + 0 \\
&= nx^{n-1}.
\end{aligned}$$

□

Solution 2, using the product rule. We prove this using the product rule and the principle of mathematical induction. For this, we need to show two things. First, we need to check the base case $n = 1$. Second, we need to show that given the formula for the n case we can derive the formula for the $n + 1$ case.

(Base case) This is a direct calculation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\, x^1 = \operatorname{st}\left(\frac{x + \Delta x - x}{\Delta x}\right) = \operatorname{st}\left(\frac{\Delta x}{\Delta x}\right) = 1.$$

(Inductive step) Assume the formula for n, i.e. that $\frac{\mathrm{d}}{\mathrm{d}x}\, x^n = nx^{n-1}$. Now use the product rule on $x^{n+1} = x^n \cdot x$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\,(x^n \cdot x) &= x^n \frac{\mathrm{d}}{\mathrm{d}x}\, x + \frac{\mathrm{d}}{\mathrm{d}x}\,(x^n) \cdot x \\ &= x^n + nx^{n-1} \cdot x \\ &= x^n + nx^n \\ &= (n+1)x^n.\end{aligned}$$

□