MATH 217M1: TAKE-HOME FINAL EXAM
DUE MONDAY 10/21

High Pass: High quality solutions for at least 6 of the 8 problems.
Pass: High quality solutions for at least 4 of the 8 problems.

Problem 1. *Suppose you divide the plane into regions by drawing (finitely many) straight lines. Prove that you can color the regions with two colors so that adjacent regions don't share a color.*

The next two problems are about square-free numbers. Say that a positive integer n is *square-free* if there are no integers $a, b > 1$ so that $n = a^2 b$.

Problem 2. *Suppose n is a positive integer. Prove that n is square-free if and only if every exponent in its prime factorization is* 1.

Problem 3. *Prove that every positive integer n can be uniquely factored as $n = rs^2$ where r and s are positive integers and r is square-free.*

The next two problems are about order isomorphisms. Recall that if $(X, \leq_X)$ and $(Y, \leq_Y)$ are orders then they are *isomorphic* if there is an isomorphism $f : X \cong Y$, where an *isomorphism* is a bijection $f : X \to Y$ with the property that $a \leq_X b$ if and only if $f(a) \leq_Y f(b)$.

Problem 4. *Prove that any two closed intervals are isomorphic as orders, where the intervals have the order from the real line. That is, if $a < b$ and $c < d$ are real numbers then the intervals $[a, b]$ and $[c, d]$ are isomorphic as orders.*

Problem 5. *Prove that the real line $\mathbb{R}$ and any open interval are isomorphic as orders. That is, if $a < b$ then $\mathbb{R}$ and (a, b) are isomorphic as orders.*

Problem 6. *Suppose $m < n$ are natural numbers. Prove that there is no injection from a set with n elements to a set with m elements.*

Problem 7. *Suppose n is a natural number and suppose that A is a set with n elements. Prove that any injection $A \to A$ is a surjection.*

Problem 8. *Define a relation $\sim$ on sets as $A \sim B$ if there is a bijection $A \to B$. Prove that $\sim$ is an equivalence relation.*