MATH 217M1: HOMEWORK 3
DUE MONDAY 9/16

High Pass: Problems 1 through 3, and three of Problems 4 through 7.
Pass: At least one of Problems 1 and 2, at least two of Problems 4 through 7, with at least four of seven problems in total.

The first few problems concern different formulations of induction. I include them here for your convenience.

- **(Least Number Principle)** Suppose $X \subseteq \mathbb{N}$. If X has an element, then X has a least element.
- **(Common Induction)** Suppose $X \subseteq \mathbb{N}$ has the properties that $0 \in X$ and for any n if $n \in X$ then $n + 1 \in X$. Then, $X = \mathbb{N}$.
- **(Strong Induction)** Suppose $X \subseteq \mathbb{N}$ has the property that for any n if every $m < n$ is in X then n is in X. Then, $X = \mathbb{N}$.

In class we did part of the proof that these principles are all equivalent—namely, we saw that the Least Number Principle implies the other two principles. The rest of the proof is homework.

Problem 1. *Prove that Common Induction implies Strong Induction.*

Problem 2. *Prove that Strong Induction implies the Least Number Principle. Explain why proving these two implications, in combination with the implications proved in class, is enough to prove all three principles equivalent.*

An idea going back to Richard Dedekind and Giuseppe Peano is that the natural numbers can be thought of as given by a recursive construction. Thus, induction becomes the defining feature of the natural numbers. Specifically, the idea is that $\mathbb{N}$ is characterized as:

- 0 is a natural number, the smallest one;
- If n is a natural number, then there is a next smallest one, called the *successor* of n and written either $s(n)$ or $n + 1$;
- $\mathbb{N}$ consists only of the numbers generated by one of the two previous rules.

Problem 3. *Given this characterization of* $\mathbb{N}$*, explain why the Common Induction principle is true.*

You can build on this characterization to define basic arithmetic operations and prove their usual properties. For example, addition $a + b$ is defined by recursion on b:

- $a + 0 = a$ for any a; and
- $a + s(b) = s(a + b)$ for any a.[1]

[1] You might prefer to write this equivalently as

$$a + (b + 1) = (a + b) + 1.$$

Problem 4. *Use the recursive definition of* $+$ *to show that* $0 + a = a$ *for any* a. [Note: You can't just use the commutativity of addition for this. The point is, you can use the recursive definition of addition to prove its basic properties. Indeed, this proposition makes the base case of the proof that addition is commutative.]

Problem 5. *Use the recursive definition of* $+$ *to show that addition is commutative:* $a + b = b + a$ *for any* a, b. [Hint: Fix a and do recursion on b.]

Problem 6. *Use the recursive definition of* $+$ *to show that addition is associative:* $a+(b+c) = (a + b) + c$. [Hint: Fix a and b and do recursion on c.]

Given the definition of $+$ on the natural numbers, you can define the order: $a \leq b$ if there is n so that $a + n = b$.

Problem 7. *Show that addition preserves order. That is, if* $a \leq b$ *then* $a + c \leq b + c$. [Hint: You can either do induction on c or do a direct proof.]