MATH 217M
A LITTLE LOGIC

The logical structure of a mathematical statement informs both how you might prove it and how you can use it in proofs. For that reason it is helpful to see a little about logic. By "mathematical statements" I mean *predicates*, statements which can be either true or false.

Examples of predicates:

- "There are infinitely many primes."
- "There exists a real-valued function which is continuous everywhere but differentiable nowhere."
- "p is prime."
- "The limit of $f(x)$ as x approaches a is L."

Some predicates have *parameters*, others do not. When we use variables to refer to a predicate we will write e.g. $P(x, y)$ to denote that the predicate P has parameters x and y. Whether $P(x, y)$ is true can depend on what you plug in for x and y!

We can analyze predicates as being built up from simpler predicates. The basic logical notions are divided into *connectives*, which connect multiple predicates to form a more complex predicate, and *quantifiers*, which make an assertion about many different mathematical objects—numbers, functions, etc. The connectives and quantifiers in a mathematical statement determine how it can be used in proofs.

Connectives

The basic connectives are *conjunction* ($\wedge$), *disjunction* ($\vee$), *negation* ($\neg$), *implication* ($\Rightarrow$), and *equivalence* ($\Leftrightarrow$).

The meaning of the connectives is given in two ways, by *truth tables* and by *inference rules*. Truth tables tell you about how the truth or falsity of the input predicates determine the truth or falsity of the compound predicate. While inference rules tell you about how the connective can be used in proofs.

Conjunction. The connective $\wedge$ corresponds to what is expressed by the English word "and". Intuitively, $P \wedge Q$ means that both P and Q are true.

You might see $P \wedge Q$ expressed in English as "P and Q" or "P, but Q".

Truth Table

P	Q	$P \wedge Q$
T	T	
T	F	
F	T	
F	F	

Observe from the symmetry in this truth table and order doesn't matter for $\wedge$: $P \wedge Q$ and $Q \wedge P$ are equivalent.

Inference rules

- From P, Q infer $P \wedge Q$.
- From $P \wedge Q$ infer P, Q.

The first inference rule tells us to prove a statement of the form $P \wedge Q$ we can separately prove P and Q. The second inference rule tells us that if we know $P \wedge Q$ then we can conclude both P and Q to use individually in our proof.

Implication. The connective $\Rightarrow$ corresponds to what is expressed by the English words "if then". Intuitively $P \Rightarrow Q$ means that if P is true then Q is also true.

There are many different ways $P \Rightarrow Q$ gets expressed in English: "if P then Q", "Q whenever P", "P is sufficient for Q", "Q is necessary for P", "Q, if P", "P implies Q", "P, hence Q", etc.

Truth Table

P	Q	$P \Rightarrow Q$
T	T	
T	F	
F	T	
F	F	

Naturally, order does matter here: $P \Rightarrow Q$ does not have the same meaning as $Q \Rightarrow P$. If the third row of this truth table seems off, hold onto that thought. We'll come back to it when we look at quantifiers. In short, this is the definition that makes things work right with quantifiers.

Inference rules

- From P, $P \Rightarrow Q$ infer Q.
- If from P you can infer Q, then you can infer $P \Rightarrow Q$.

The first inference rule is known as *modus ponens*, and is how $P \Rightarrow Q$ is most often used in proofs. Namely, if you know both P and $P \Rightarrow Q$ then you can conclude Q. For example, you might know both that n^2 is even and that if n^2 is even then n is even. *Modus ponens* says you can conclude n is even.

The second rule tells you how you can prove a statement of the form $P \Rightarrow Q$. It says that if you can prove Q from an assumption of P then this lets you conclude $P \Rightarrow Q$.

Equivalence. The connective $\Leftrightarrow$ corresponds to what is expressed by the English words "if and only if". This phrase is commonly used by mathematicians, so the abbreviation "iff" gets used. Intuitively, $P \Leftrightarrow Q$ means that P and Q are equivalent.

Like implication, there are many different ways $P \Leftrightarrow Q$ gets expressed in English: "P if and only if Q", "P and Q are equivalent", and "P is a necessary and sufficient condition for Q" are just some of them.

Truth Table

P	Q	$P \Leftrightarrow Q$
T	T	
T	F	
F	T	
F	F	

Note the symmetry; order doesn't matter.

Inference rules

- From $P \Leftrightarrow Q$ infer $P \Rightarrow Q$, $Q \Rightarrow P$.
- From $P \Rightarrow Q$, $Q \Rightarrow P$ infer $P \Leftrightarrow Q$.

These rules express that $P \Leftrightarrow Q$ has the same meaning as "$P \Rightarrow Q$ and $Q \Rightarrow P$". To prove an equivalence you have two implications to prove. And if you know an equivalence then you may conclude either of the implications.

Negation. The connective $\neg$ corresponds to what is expressed by the English word "not". Intuitively, $\neg P$ means that P is not true.

English phrases using the word "not" usually correspond to a statement of the form $\neg P$. But be aware that the word "not" can appear in many different places in a sentence: "it is not the case that there is a largest integer", "n is not prime", "X is infinite [= not finite]".

Truth Table

P	$\neg P$
T	
F	

For the inference rules let's introduce a new symbol: $\perp$ is a special symbol meaning an impossibility, e.g. something like $P \wedge \neg P$.

Inference rules

- From $P \Rightarrow \perp$ infer $\neg P$.
- From $\neg\neg P$ infer P.
- From P infer $\neg\neg P$.

The first inference rule expresses proof by contradiction: to prove $\neg P$ you show that assuming P implies an impossibility.

The second and third inference rules express the principle of *double negation*, which says that any predicate is equivalent to its double negation.

We can also now express two new rules about implication.

- From $\neg Q$, $P \Rightarrow Q$ infer $\neg P$.
- If from $\neg Q$ you can infer $\neg P$, then you can infer $P \Rightarrow Q$.

These rules both come from the fact that $P \Rightarrow Q$ is equivalent to its *contrapositive* $\neg Q \Rightarrow \neg P$. The first rule is called *modus tollens* and the second expresses what is called *proof by contrapositive.*

Disjunction. The connective $\vee$ corresponds roughly to what is expressed by the English word "or". Intuitively, $P \wedge Q$ means that at least one of P and Q is true.

Note that this is an 'inclusive or', not an 'exclusive or', allowing both options at once. When a waiter tells you the entree comes with soup or with a side salad, then both is not an acceptable answer. But when a mathematician tells you that a number is either ≥ 0 or ≤ 0, then both is an acceptable answer.

You might see $P \vee Q$ expressed in English as "P or Q", "at least one of P and Q", "either of P or Q", or so on.

Truth Table

P	Q	$P \vee Q$
T	T	
T	F	
F	T	
F	F	

Note the symmetry; order doesn't matter.

Inference rules

- From P infer $P \vee Q$.
- From $P \Rightarrow R$, $Q \Rightarrow R$ infer $(P \vee Q) \Rightarrow R$.
- From $\neg P \Rightarrow Q$ infer $P \vee Q$.

The first rule expresses one way to prove a disjunction: you just need to prove one of the two disjuncts is true.

The second rule expresses proof by cases. It expresses that if you can draw the same conclusion from either P or Q, then you can conclude that $P \vee Q$ implies that conclusion.

The third rule expresses another way to prove a disjunction. Namely, what do you do if you can't settle for sure which of P or Q is true? In this case you want to assume one is false and prove the other is true.

Other connectives

You might worry that these connectives aren't expressive enough. What if, for example, you want to express an exclusive or: either P or Q, but not both. You can see why the logical formula you get from that English definition gives exclusive or by comparing truth tables:

P	Q	P xor Q	
T	T		
T	F		
F	T		
F	F		

In a similar fashion, any possible connective you might want to define can be expressed in terms of the five basic logical connectives. (Indeed, it turns out you can get away with fewer.)

Yet more inference rules

You can derive yet more inference rules by looking at truth tables. In particular, two logical expressions are equivalent if they have the same truth table.

An important example is given by DeMorgan's laws:

- $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$;
- $\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$;

Fill out the truth tables to confirm equivalence:

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T		
T	F		
F	T		
F	F		

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T		
T	F		
F	T		
F	F		

Here are some more equivalences you can check using truth tables:

- *Associativity of* $\wedge$*:* $P \wedge (Q \wedge R)$ is equivalent to $(P \wedge Q) \wedge R$.
- *Associativity of* $\vee$*:* $P \vee (Q \vee R)$ is equivalent to $(P \vee Q) \vee R$.
- *Distributivity of* $\wedge$ *over* $\vee$*:* $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.
- *Distributivity of* $\vee$ *over* $\wedge$*:* $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$.

False inference rules

There are some invalid inferences that people sometimes make. Let me highlight a couple common ones.

- *Affirming the consequent:* From Q, $P \Rightarrow Q$ infer P.
- *Denying the antecedent:* From $\neg P$, $P \Rightarrow Q$ infer $\neg Q$.

The fallacy in each is replacing the implication $P \Rightarrow Q$ with its *converse* $Q \Rightarrow P$. An implication is not equivalent to its converse!

Predicates and sets

There is a close connection between logic and set theory.

- Work over a domain U. For example, if you're dealing with natural numbers you'd take $U = \mathbb{N}$.
- If $P(x)$ is a predicate about objects x from U, then the *extension* of $P(x)$ is the set $P = \{x \in U : P(x)\}$. In words, the extension is the set of all x for which $P(x)$ is true. This notation for defining a set is called *set builder notation*. It amounts to giving a predicate which defines the objects in the set.
- The connectives $\wedge$, $\vee$, and $\neg$ correspond to set theoretic operations.
  - The extension of $P(x) \wedge Q(x)$ is the *intersection* $P \cap Q$.
  - The extension of $P(x) \vee Q(x)$ is the *union* $P \cup Q$.
  - The extension of $\neg P(x)$ is the *complement* $U \setminus P$.
- The connectives $\Rightarrow$ and $\Leftrightarrow$ correspond to statements about *subsets* and set equality.
  - $P(x) \Rightarrow Q(x)$ is true for all $x \in U$ if and only if P is a *subset* of Q, written $P \subseteq Q$.
  - $P(x) \Leftrightarrow Q(x)$ is true for all $x \in U$ if and only if $P = Q$.
  - Note that $P = Q$ is equivalent to $(P \subseteq Q) \wedge (Q \subseteq P)$.

You can graphically express these relationships with *Venn diagrams*.

Quantifiers

Mathematics is full of statements like "every non-constant polynomial has a root in the complex numbers". That is, we aren't just interested in a predicate $P(f)$ like "f has a root in the complex numbers" by itself. We want to know something about the fs' behavior as a whole.

You can rewrite this expression to make the *quantifiers* clear: "*for all* non-constant polynomials f *there is* a complex number r so that $f(r) = 0$".

- "For all" is the *universal quantifier*. We write $\forall x\ P(x)$ for "for all x, $P(x)$".
- "There is" is the *existential quantifier*. We write $\exists x\ P(x)$ for "there is x so that $P(x)$".

Usually you don't want to look at all objects in the universe of mathematics. Instead, you want to only look at certain kinds of objects. For this we use *bounded quantifiers*, restricting the objects looked at to a certain set.

Suppose we want to quantify over a smaller domain D.

- $\forall x \in D\ P(x)$ is equivalent to $\forall x\ (x \in D \Rightarrow P(x))$.
- $\exists x \in D\ P(x)$ is equivalent to $\exists x\ (x \in D \wedge P(x))$.

You also sometimes see bounded quantifiers for ways of restricting beyond set membership. For example,

$$\forall x \geq 0\ \exists y \in \mathbb{R}\ x = y^2$$

expresses the fact that ever real ≥ 0 is a perfect square.

You can write down inference rules explaining how quantifiers are used in proofs. I'll write these for bounded quantifiers, as those are the most commonly used. Here, x is a variable for an object in a domain D and t is a *term* or name for a specific object in D.

- If you can infer $P(x)$ about arbitrary $x \in D$ then you can infer $\forall x \in D\ P(x)$.
- If you can infer $P(t)$ then you can infer $\exists x \in D\ P(x)$.

These rules express how to prove statements about quantifiers. To prove a statement about all $x \in D$, you assume you are looking at some random $x \in D$, where you're not allowed to make any further assumptions. And then you want to concluded $P(x)$ for this arbitrary object. Whereas to prove the existential statement it suffices to look at a specific example t.

- From $\forall x \in D\ P(x)$ infer $P(t)$.
- From $\exists x \in D\ P(x)$ infer $P(x)$, where x is an unused variable.

These rules express how you can use known statements about quantifiers. If you know something is true for all $x \in D$, then you can infer its true for any specific t you care about. Whereas if you only know that it's true for some $x \in D$, then all you can do is give a name to the witnessing object.

We also have rules which express how to rewrite statements about quantifiers. The most important of these are DeMorgan's laws.

- $\neg\forall x\ P(x)$ is equivalent to $\exists x\ \neg P(x)$.
- $\neg\exists x\ P(x)$ is equivalent to $\forall x\ \neg P(x)$.

These are useful for proofs by contradiction. For example, if you want to prove $\forall x\ P(x)$ note it's equivalent to $\neg\exists x\ \neg P(x)$. This is an appropriate form to use proof by contradiction.

To explain the name: you can think of $\forall x\ P(x)$ as a big, possibly infinite, conjunction over all x's, and $\exists x\ P(x)$ as a big, possibly infinite, disjunction. Then these equivalences about quantifiers exactly correspond to the DeMorgan's laws about connectives.

A note on writing mathematics and logical symbols

The various logical symbols we've seen this week are great for studying logic. Much like it's nicer to write "2 + 3" than "the sum of 2 and 3", when proving results about logic it's nicer to use symbols. These symbols also are commonly used by mathematicians as abbreviations, especially when writing on chalkboards.

These symbols are also great to use when breaking down the logical structure of a statement. Whenever you want to prove a statement, if you don't know how to begin a good first step on your scratch paper is to write the statement in logical form.

That said, it's considered bad writing to use these symbols as abbreviations in mathematical writing. For example, it's preferred to write "every differentiable function is continuous" instead of "$\forall f$, f is differentiable $\Rightarrow$ f is continuous". An overuse of logical symbols can make your arguments harder for a reader to follow.

In general, the overuse of symbols can make mathematical writing more opaque. A good question to ask yourself is, am I using these symbols because the brevity they provide makes the argument more clear? Or am I just doing it because I don't want to write out a phrase in English?

Exercises

High Pass: All five problems.
Pass: At least two parts of problem 1, at least three of the remaining problems.

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(1) Translate the following mathematical statements into logical notation. For the first two please use the order relation $<$ on $\mathbb{N}$ as your basic mathematical concept. For the last one, use the order relation $<$ on $\mathbb{R}$, the absolute value function $|x|$, and the arithmetic operations as your basic mathematical concepts. That is, besides the logical symbols you should only use the prescribed concepts and set membership $\in$.
   - The *least number principle*, i.e. if $X \subseteq \mathbb{N}$ has an element then it has a least element.
   - *Strong induction*, i.e. if $X \subseteq \mathbb{N}$ has the property that for any $n \in \mathbb{N}$ if every $m < n$ is in X then $n \in X$, then $X = \mathbb{N}$.
   - The *epsilon-delta* definition of continuity at a point a: for any $\varepsilon > 0$ there is $\delta > 0$ so that if the distance from x to a is less than δ then the distance from $f(x)$ to $f(a)$ is less than ϵ.

(2) Check the distributivity laws for $\wedge$ and $\vee$ using truth tables.

(3) Consider the predicate $B(x, y)$ expressing "x is bigger than y". Use this predicate to give examples explaining why the order of quantifiers matters: $\forall x\, \exists y\, P(x, y)$ is not equivalent to $\exists y\, \forall x\, P(x, y)$.

(4) Let A be the set of all prime numbers > 2 and let B be the set of odd numbers. Write definitions of A and B in set builder notation and prove that $A \subseteq B$. *[Hint: This is asking you to prove an if-then statement.]*

(5) Let A be the set of all nonnegative real numbers and let B be the set of all squares x^2 of a real number x. Write definitions of A and B in set builder notation and prove that $A = B$. *[Hint: this is asking you to prove an if-and-only-if statement.]*