MATH 217M
RELATIONS AND FUNCTIONS

Mathematics isn't just about individual objects, it's about how objects are related to each other and what you can do with them. We can think of *relations* and *functions* as themselves objects of mathematical study, and they are a key element of many branches of math.

Relations

A *relation* is a way two or more objects can be related. Most relations in mathematics are *binary*, relating two objects.

Two ways to think about a relation $\star$:

- Its *intension*, how it is defined.
- Its *extension*, the set of objects related by $\star$.

The standard in mathematics is to take the extension as *the* definition of a relation.

- A *binary relation* on a set X is a subset of X^2, the set of ordered pairs (x, y) from X.
- Often people drop the word "binary" and just say "relation".
- If $\star$ is a relation on X then $x \star y$ just means $(x, y) \in \star$.
- You can also talk about a relation from a set X to a set Y, but that shows up less in practice.

Here are a few important properties a relation can have. Here let $\star$ be a relation on X.

- $\star$ is *reflexive* if $x \star x$ for every $x \in X$.
- $\star$ is *symmetric* if $x \star y$ implies $y \star x$ for every $x, y \in X$.
- $\star$ is *transitive* if $x \star y$ and $y \star z$ implies $x \star z$ for every $x, y, z \in X$.

Relations that satisfy all three properties are particularly interesting.

- A relation $\sim$ on X is an *equivalence relation* if $\sim$ is reflexive, symmetric, and transitive.
- Equivalence relations enjoy a lot of the same properties as the equality relation $=$.
- They are used to formalize a notion that two objects are the same.

Equivalence relations are tightly connected to partitions of a set.

- If $\sim$ is an equivalence relation on X and $x \in X$ then the *$\sim$-equivalence class for x* is
$$[x] = [x]_\sim = \{y \in X : x \sim y\}.$$
- A *partition* of X is a family of nonempty, pairwise disjoint subsets of X whose union is all of X.
- When $\sim$ is an equivalence relation the $\sim$-equivalence classes form a partition of X.
- Notation: we write $X/\sim$ for the family of $\sim$-equivalence classes.
- If $\Pi = \{X_0, X_1, \dots\}$ is a partition of X it yields an equivalence relation $\sim_\Pi$ defined as $x \sim_\Pi y$ if x and y are in the same piece of Π.

Functions

A *function* is a way of assigning input objects to outputs. We write $f : X \to Y$ to say that f is a function sending inputs from X to outputs in Y.

Two ways to think about a function f:

- Its *intension*, how it is defined.
- Its *extension*, the set of input-output pairs.

The standard in mathematics is to take the extension as *the* definition of a relation.

- A *function* $f : X \to Y$ from a set X to a set Y is a set of ordered pairs $(x, f(x))$ so that for each $x \in X$ there is a unique $f(x) \in Y$.
- Note that requiring $f(x)$ to be unique expresses the content of the virtual line test.
- X is called the *domain* of f, written f, and Y is called the *codomain*.
- The *range* of f, written f, is the set outputs of f, namely the set of $y \in Y$ so that $y = f(x)$ for some $x \in X$.

This definition can be modified in a few ways, and these modifications also get used in mathematics.

- Some mathematicians make the codomain part of the definition, so that e.g. $(\,\cdot\,)^2 : \mathbb{Z} \to \mathbb{Z}$ and $(\,\cdot\,)^2 : \mathbb{Z} \to \mathbb{N}$ are considered different functions.
- Instead of requiring there to be $f(x)$ for every x in X we could just ask it exists for some. We call this a *partial function* from X to Y.
- Instead of requiring $f(x)$ to be unique we could allow multiple outputs for one input. We call this a *multi-valued function* from X to Y.

We can also handle functions with multiple inputs, by thinking of the domain as a set of ordered pairs or ordered tuples.

- If $f : X^2 \to Y$ we write e.g. $f(x_0, x_1)$ rather than $f((x_0, x_1))$. And similarly for higher arity functions.

Here's a few important properties a function can have. Let $f : X \to Y$ be a function.

- If $f = Y$ we say that f is *onto* Y or *surjective onto* Y. When the codomain is clear we usually just say onto or surjective.
- If $f(x) \neq f(x')$ whenever $x \neq x'$ are distinct inputs from X we say that f is *one-to-one* or *injective*. Equivalently, f is one-to-one if $f(x) = f(x')$ implies $x = x'$.
- If f is both one-to-one and onto Y we say that f is a *bijection* onto Y. In older books you might see this called a *one-to-one correspondence between* X *and* Y.

Here are some basic definitions about functions.

- The *composition* of $f : X \to Y$ and $g : Y \to Z$ is the function $g \circ f : X \to Z$ defined as $(g \circ f)(x) = g(f(x))$.
- If $f : X \to Y$ is one-to-one then we can define its *inverse* $f^{-1} : f \to X$ as $x = f^{-1}(y)$ iff $y = f(x)$.
- The *identity function* on a set X is the function $\mathrm{id} : X \to X$ defined as $\mathrm{id}(x) = x$.

Here are some facts about these definitions.

- Let $f : X \to Y$ then $f \circ \mathrm{id} = \mathrm{id} \circ f = f$.
- If $f : X \to Y$ and $g : Y \to Z$ are both surjections then $g \circ f : X \to Z$ is also a surjection.

- If $f : X \to Y$ and $g : Y \to Z$ are both injections then $g \circ f : X \to Z$ is also a injection.
- If $f : X \to Y$ and $g : Y \to Z$ are both bijections then $g \circ f : X \to Z$ is also a bijection.
- If $f : X \to Y$ is an injection then $f^{-1} : f \to X$ is a surjection.
- If $f : X \to Y$ is a bijection then $f^{-1} : Y \to X$ is also a bijection.

Functions give rise to equivalence relations.

- If $f : X \to Y$ is a function define a relation $\sim_f$ on X as $x \sim_f x'$ if $f(x) = f(x')$. Then $\sim_f$ is always an equivalence relation.
- If $\sim$ is an equivalence relation on X then $\sim = \sim_f$ where $f : X \to X/\sim$ is the function $f(x) = [x]_\sim$.

Often mathematicians want to know whether a function and an equivalence relation play nicely together. Here let $f : X \to X$ and $\sim$ be an equivalence relation on X.

- Say that f is *well-defined on $\sim$-equivalence classes* if $x \sim y$ implies $f(x) \sim f(y)$.
- An analogous definition works if f has more than one input.
- When this happens, we can think of f as a function on the equivalence classes $X/\sim$, by setting $f([x]_\sim) = [f(x)]_\sim$.