# MATH 218M TECHNICAL CONTENT
# BOOLEAN LOGIC

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## 1. Introduction

Logic is sometimes referred to the science of correct thought. One can and should question just what that means, but it's a starting point for understanding what the discipline is about. In practice much of the study of logic is the study of various formal systems used to model "correct" reasoning. Modern work in logic is characterized by the vast diversity of formal systems, and their application has been fruitful in mathematics, computer science, philosophy, and linguistics.

In this class we will survey three logical systems. The first is boolean logic. (Some sources would instead call this classical propositional logic; logic suffers from many names for the same thing.) This system is the "standard" logical system one uses in e.g. mathematics. Even those who think classical logic is flawed or harmful will contrast their alternative logics to it as a baseline.

We will follow the same approach. Boolean logic will be our binary baseline against which we will compare the other logics we study, and thus we will survey it first. Later we will study fuzzy logic and intuitionistic logic. Briefly, fuzzy logic contrasts to boolean logic by holding that things are more nuanced than true versus false, while intuitionistic logic contrasts by being more stringent as what counts as sufficient evidence.

## 2. The language of logic

The basic language consists of *propositional variables* and *constants*, and *connectives* which combine simpler logical statements into a more complex statement.

To use some jargon, this is the language of *propositional logic*. We will later extend this to the language of *predicate logic*, synonymously *first-order logic*.

For *propositional variables* we will use letters like $P, Q, \ldots$. In a calculus class you'll have seen variables like $x$ or $f$ standing in for real numbers or functions of real numbers. Our variables will stand in for *propositions*. Roughly speaking, a proposition is a statement which can be given a truth value. So "John is a man" is a proposition but "Is John a man?" is not nor is "Be a man".

The *propositional constants* are the possible *truth values* of a proposition. For boolean logic there are two constants: 0 for false and 1 for true. (Some sources will instead use $F$ and $T$ or $\bot$ and $\top$.)

We will take as basic four connectives. (Other sources may have more or fewer.) They are AND $\wedge$, OR $\vee$, NOT $\neg$, and IF-THEN $\Rightarrow$.

From these elements of the language we build up *formulas*. As meta-syntatic variables to refer to formulas we will use Greek letters like $\varphi$ ("phi") or $\psi$ ("psi").

**Definition 1.** The *formulas* of propositional logic are defined inductively:

- 0 and 1 are formulas;

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- Any propositional variable $P$ is a formula;
- If $\varphi$ and $\psi$ are formulas then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi$, and $(\varphi \Rightarrow \psi)$.
- Nothing else is a formula.

The point of the parentheses is to make clear the order of operations. For example, $(P \wedge Q) \vee R$ and $P \wedge (Q \vee R)$ have a different meaning. When writing formulas we will drop parentheses when the meaning is clear.

*Exercise* 1. Which of the following are formulas? If something is not a formula, can you make it one by inserting parentheses in appropriate places? Does it matter where you place the parentheses?

$$\begin{array}{cccc} 0 \Rightarrow 1 & P \wedge Q \wedge R \wedge S & \neg \vee P & \neg\neg Q \\ 0 \neg P & (P \Rightarrow Q) \wedge (Q \Rightarrow P) & P \wedge Q \vee R & P \Rightarrow Q \vee R \end{array}$$

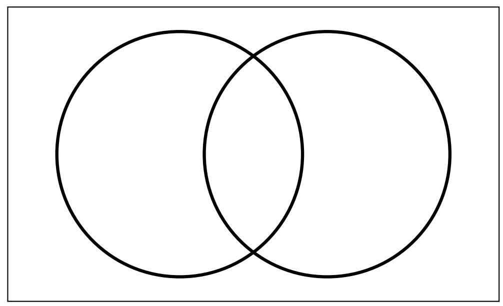
## 3. The meaning of the connectives

In boolean logic the meanings of the connectives can be explained by saying how the truth values of the inputs determine the truth value of the compound statement. A short way to represent this is through the use of *truth tables*. It can also be expressed visually through *Venn diagrams*.

A *truth table* consists of one row for each possible combination of truth values of the inputs, and gives the truth value of the compound statement. A *Venn diagram* consists of overlapping circles with the shading of each region expressing the possible truth values for the compound statement.

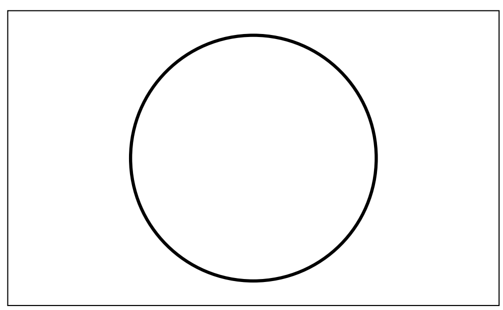
**AND (Conjunction).** In informal English, $\varphi \wedge \psi$ means "$\varphi$ and $\psi$". You might see it expressed in other ways: "$\varphi$ but $\psi$", "each of $\varphi$, $\psi$", etc. However it is written, it express the idea that you're asking for both conjuncts to be true.

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| 1 | 1 | |
| 1 | 0 | |
| 0 | 1 | |
| 0 | 0 | |



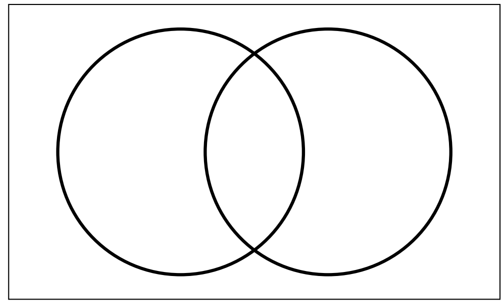
**NOT (Negation).** In informal English, $\neg\varphi$ means "not $\varphi$". It's ungrammatical to just put "not" at the front of a sentence so instead it gets expressed as e.g. "John is not a man" or "$\sqrt{2}$ is irrational". However it's written, it expresses negation.

| $P$ | $\neg P$ |
|---|---|
| 1 | |
| 0 | |



**OR (Disjunction).** In informal English, $\varphi \vee \psi$ means "$\varphi$ or $\psi$". This "or" is an *inclusive* or, meaning that we want to allow the possibility that both inputs are true. This is at odds with how the word is sometimes used in ordinary language. For example, if a waiter tells you that your hamburger comes with fries or a side salad it is not acceptable to answer that you want both. This connective expresses that at least one of the disjuncts is true.

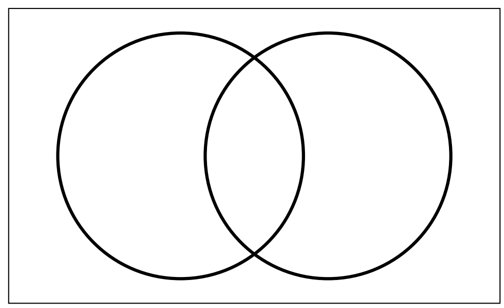
| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| 1 | 1 | |
| 1 | 0 | |
| 0 | 1 | |
| 0 | 0 | |



**IF-THEN (Material conditional).** This connective, also called the *material conditional* is the most controversial as to its connection to informal English. Very roughly $\varphi \Rightarrow \psi$ means "if $\varphi$, then $\psi$".

This explains how to fill in some of the rows in the table: "if true then true" should be true and "if true then false" should be false. But it doesn't seem to express how to fill in the rows where the antecedent $\varphi$ is false. For boolean logic, connectives are *truth functional*—the output depends only on the input truth values, not what they mean. So to express that if the antecedent is false then it doesn't determine the consequent one way or the other, we say the bottom two rows are true.

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| 1 | 1 | |
| 1 | 0 | |
| 0 | 1 | |
| 0 | 0 | |



We will come back to this point when we study intuitionistic logic, but let's briefly discuss why this connective is controversial. The meaning of $\Rightarrow$ as expressed by the above truth table would say that the sentence "if the moon is made of cheese then $2 + 2 = 4$" would be true.[1] Yet the sentence appears nonsensical—whatever substance makes up the moon it has nothing to do with arithmetic facts.

For another example, later we will see that $(P \Rightarrow Q) \vee (Q \Rightarrow P)$ is always true. Yet in ordinary language if we grab two random assertions there's no reason to believe either implies the other. We wouldn't say that either of "if it rains tomorrow then Harris will win the election" or "if Harris wins the election then it will rain tomorrow" is true.

*Exercise* 2. Fill in all truth tables and venn diagrams from this section.

*Exercise* 3. For each of the following sentences rewrite it in the form of a logical formula, replacing atomic utterances with propositional variables. State what each variable stands for.

- If $k$ is a multiple of 5 then the last digit of $k$ is either a 0 or a 5.
- You can have the soup or the salad but you cannot have both.

[1] This example comes from the philosopher C. I. Lewis.

- Alice is an adult, female human.
- Either if it rains tomorrow then Harris will win the election, or if Harris wins the election then it will rain tomorrow.
- Either Harris or Trump will win the election.

As an alternative to truth tables, you could've instead defined the meaning of the connectives using arithmetic ideas.

*Exercise* 4. For the following suppose that $p$ and $q$ are real numbers which are either 0 or 1.

- Check that the truth value of $\neg p$ is the same as $1 - p$.
- Check that the truth value of $p \vee q$ is the same as $\max(p, q)$.
- Check that the truth value of $p \wedge q$ is the same as $\min(p, q)$ and the same as $p \cdot q$.
- Check that the truth value of $p \Rightarrow q$ is 1 if and only if $p \leq q$.

## 4. Valuations

As the German philosopher Immanuel Kant put it, "the old and famous question, which drives logician into a corner where they either have to involve themselves in a wretched dialogue or to confess their ignorance, and hence the vanity of their entire art, is this: what is truth?" We will not attempt a full answer to this difficult question. But we will answer an easier question: given a formula in boolean logic how do we analyze its truth?

If the formula has no variables then the truth tables for the connectives give us an algorithm for determining the truth value of a formula. Namely, repeatedly apply the rules encoded in the truth tables starting at the innermost level until you get a truth value for the whole formula.

*Exercise* 5. Evaluate the truth values of the formulas

- $(0 \Rightarrow 1) \wedge (1 \Rightarrow 1) \wedge (0 \Rightarrow 0)$
- $(1 \vee 0) \wedge \neg 1$

For formulas with variables this doesn't work—how can you carry out this procedure if you don't know the values of the variables? As such, it doesn't quite make sense to talk about the truth value of an arbitrary formula. Compare to how in mathematics it doesn't make sense to say $\frac{n}{3}$ is a whole number unless you know something about what $n$ stands for. However, if you choose some truth values of the variables then the truth value of the formula is determined, by following the meaning of the connectives.

**Definition 2.** A *valuation* $v$ is an assignment of truth values to variables. For a formula $\varphi$ we say a *valuation for* $\varphi$ is any valuation which assigns truth values to every variable which appears in $\varphi$. (It could also assign values to variables which don't appear.)

We write valuations like with function notation: $v(P) = 1$ means that $v$ assigns $P$ the truth value 1. We can expand this to assign truth values to formulas by following the rules encoded in the truth tables.

**Definition 3.** Let $v$ be a valuation for a formula $\varphi$. We define $v(\varphi)$, the *evaluation map* which gives the truth value of a formula, as follows:

- $v(0) = 0$ and $v(1) = 1$;
- $v(\varphi \wedge \psi) = 1$ if $v(\varphi) = 1$ and $v(\psi) = 1$, and is 0 otherwise;
- $v(\varphi \vee \psi) = 1$ if either $v(\varphi)$ or $v(\psi)$ is 1, and is 0 otherwise;
- $v(\neg\varphi) = 1$ if $v(\varphi) = 0$, and is 0 otherwise;
- $v(\varphi \Rightarrow \psi) = 1$ if either $v(\varphi) = 0$ or $v(\psi) = 1$, and is 0 otherwise.

Noting that $v(\varphi)$ is always a truth value you might instead have made the definitions using the connectives themselves. For example, $v(\varphi \Rightarrow \psi)$ is the truth value of $v(\varphi) \Rightarrow v(\psi)$.

**Observation 4.** *Let $v$ and $w$ be two evaluations for $\varphi$. If $v$ and $w$ agree on the truth value of variables which occur in $\varphi$, then $v(\varphi) = w(\varphi)$. In other words, what a valuation says about variables which aren't in $\varphi$ has no impact on how it evaluates $\varphi$.* □

A valuation $v$ along with its evaluation $v(\varphi)$ of a formula $\varphi$ corresponds to a row in a truth table. So you can think of truth tables as listing all possible valuations for a formula.

**Definition 5.** Two formulas $\varphi$ and $\psi$ are *logically equivalent* if they contain the same variables and given any valuation $v$ for $\varphi$ and $\psi$ we have $v(\varphi) = v(\psi)$. We write $\varphi \equiv \psi$.

In terms of truth tables, this says that two formulas are logically equivalent if they have the same truth tables.

Now we have a way to express that two formulas have the same meaning. We are in a position to justify some facts that should be intuitive from the meanings of the connectives.

**Fact 6.** *$\wedge$ and $\vee$ and both* commutative *and* associative*. That is,*

$$P \wedge Q \equiv Q \wedge P \qquad \textit{and} \qquad (P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R),$$

*and similarly for $\vee$.*

To check this fact, we simply have to fill out truth tables for each pair of formulas and check they're the same.

| $P$ | $Q$ | $P \wedge Q$ | $Q \wedge P$ |
|---|---|---|---|
| 1 | 1 | | |
| 1 | 0 | | |
| 0 | 1 | | |
| 0 | 0 | | |

| $P$ | $Q$ | $R$ | $(P \wedge Q) \wedge R$ | $P \wedge (Q \wedge R)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | | |
| 1 | 1 | 0 | | |
| 1 | 0 | 1 | | |
| 1 | 0 | 0 | | |
| 0 | 1 | 1 | | |
| 0 | 1 | 0 | | |
| 0 | 0 | 1 | | |
| 0 | 0 | 0 | | |

*Exercise* 6. Check the commutativity and associativity laws for $\vee$ by filling out the appropriate truth tables

This method is very powerful. Let's use it to see some more equivalences.

**Fact 7** (Distributivity laws)**.** *The following pairs of formulas are equivalent:*

$$\begin{aligned} P \vee (Q \wedge R) &\equiv (P \vee Q) \wedge (P \vee R) \\ P \wedge (Q \vee R) &\equiv (P \wedge Q) \vee (P \wedge R) \end{aligned}$$

**Fact 8** (DeMorgan's laws)**.** *The following pairs of formulas are equivalent:*

$$\begin{array}{ccc} \neg(P \wedge Q) & \equiv & \neg P \vee \neg Q \\ \neg(P \vee Q) & \equiv & \neg P \wedge \neg Q \end{array}$$

| $P$ | $Q$ | $\neg(P \wedge Q)$ | $\neg P \vee \neg Q$ |
|---|---|---|---|
| 1 | 1 | | |
| 1 | 0 | | |
| 0 | 1 | | |
| 0 | 0 | | |

| $P$ | $Q$ | $\neg(P \vee Q)$ | $\neg P \wedge \neg Q$ |
|---|---|---|---|
| 1 | 1 | | |
| 1 | 0 | | |
| 0 | 1 | | |
| 0 | 0 | | |

Valuations allow us to define other notions.

**Definition 9.** A formula $\varphi$ is *satisfiable* if there is some valuation $v$ so that $v(\varphi) = 1$, otherwise $\varphi$ is *unsatisfiable*. And we say that $\varphi$ is a *tautology* if $v(\varphi) = 1$ for every valuation $v$.

In terms of truth tables, satisfiable means at least one row has a truth value of 1, unsatisfiable means every row has a truth value of 0, and tautological means every row has a truth value of 1.

*Exercise* 7. For each of the following formulas determine whether it is a tautology, satisfiable but not tautological, or unsatisfiable.

- $P \wedge \neg P$
- $P \Rightarrow P$
- $P \Rightarrow Q \vee Q \Rightarrow P$

Truth tables provide us an important meta-theoretical fact about boolean logic.

**Theorem 10.** *For each of the following there is algorithm to determine the answer.*

- *Whether two formulas are logically equivalent;*
- *Whether a formula is satisfiable;*
- *Whether a formula is a tautology.*

In each case, the algorithm is to fill out the truth table for the formula(s) and then check what it tells you. Note however that this algorithm is rather slow. If a formula has $n$ many variables then its truth table will have $2^n$ rows. For example, a 5 variable formula will have $2^5 = 32$ rows to fill out. This is doable but tedious by hand. Double the number of variables and you have $2^{10} = 1024$ rows. A computer could do this, but double the number of variables again and that's $2^{20} \approx 1$ million rows.

*Exercise* 8. Show that $P$ is logically equivalent to $\neg\neg P$.

*Exercise* 9. Show that $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$. What does this say about the meaning of the connective $\Rightarrow$?

## 5. What if we expand the language?

The four connectives seem like reasonable choices to include in our formal language, but are they enough?

A hint at the answer is found in the last exercise in the previous section. You checked that $P \Rightarrow Q$ was logically equivalent to $\neg P \vee Q$. What this means is, if you didn't already have the connective $\Rightarrow$ you could've defined it using $\neg$ and $\vee$.

In a similar manner we could introduce new connectives to our quartet. Two of the most commonly used are IFF $\Leftrightarrow$ and XOR $\oplus$. "Iff" is the usual abbreviation for "if and only if" expressing that an if-then relationship goes both ways. Meanwhile xor is the exclusive or—this or that but not both.

**Definition 11.** $P \Leftrightarrow Q$ is defined as $P \Rightarrow Q \wedge Q \Rightarrow P$. And $P \oplus Q$ is defined as $(P \vee Q) \wedge \neg(P \wedge Q)$.

*Exercise* 10. Write the truth tables for $P \Leftrightarrow Q$ and $P \oplus Q$.

*Remark* 12. If you've seen some modern algebra, there's a connection to the finite field $\mathbb{F}_2$. Recall that this is the unique up to isomorphism two element field, which can be thought of as the equivalence classes of integers modulo 2. For another way to see it: the boolean truth values 0 and 1 can be thought of as the elements of $\mathbb{F}_2$, with multiplication being given by $\wedge$ and addition being given by $\oplus$.

Are there limits to what we can express with our four connectives? Any possible connective we might define has its meaning fixed by its truth table. So what we're really asking is, can we realize any truth table pattern with our four connectives?

For *unary* connectives, namely those with one input, the corresponding truth table has only two rows. Thus, there's $2^2 = 4$ possible unary connectives. It's a small bit of work to check that the four possibilities are realized by the definitions $P$, $\neg P$, 0, and 1. Here, by 0 and 1 I mean the connective that always has 0 (respectively 1) as its output, no matter what the input is. (Compare to the constant function $y = 7$ on the real line versus the number 7.)

*Exercise* 11. Write down the truth table for these four, each considered as a unary connective with the single input $P$.

Looking next at *binary* connectives—those with two inputs—the truth tables have four rows, for $2^4 = 16$ possible connectives. And if we looked at connectives with three inputs we'd have $2^8 = 256$ possibilities. This is too many to do one at a time, so let's look for a general idea.

When you're looking for a general pattern, a good place to start is with an example.

*Exercise* 12. Imagine the two binary connectives $\star$ and $\dagger$ given by the following two truth tables. Write a definition of each connective in terms of $\wedge$, $\vee$, and $\neg$. Can you generalize your process to work for any truth table?

| $P$ | $Q$ | $P \star Q$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 0 |

| $P$ | $Q$ | $P \dagger Q$ |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

*Attempt the previous exercise first before continuing reading!*

There's a few approaches one might take, but the following is one that generalizes. Each row of the truth table corresponds to a certain pattern of truth values for $P$ and $Q$. We can identify a row using $\neg$ and $\wedge$. Namely:

- $P \wedge Q$ is the $1, 1$ row;
- $P \wedge \neg Q$ is the $1, 0$ row;
- $\neg P \wedge \neg Q$ is the $0, 1$ row; and
- $\neg P \wedge \neg Q$ is the $0, 0$ row.

To give a formula defining a truth table we write down the $\neg, \wedge$-formulas for each row with a truth value of 1 and then combine them together with $\vee$s.

This procedure always gives a formula, but it usually won't be the shorted formula you could've found. For example, for the truth table that's always 1s this method would give the definition

$$(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

instead of the much shorter 1.

The same idea works for a truth table with $n$ variables. A conjunction of variables or negated variables identifies each row, and you take the disjunction for each row with a truth value of 1. Altogether, this gives us the following fact.

**Definition 13.** A set of connectives is called *complete* if any truth table can be realized by a formula using those connectives.

**Theorem 14.** *The connectives $\wedge$, $\vee$, $\neg$, and $\Rightarrow$ form a complete set of connectives.* $\square$

Earlier we saw how to define $\Rightarrow$ in terms of $\vee$ and $\neg$, so in fact we have that $\wedge$, $\vee$, and $\neg$ form a complete set of connectives. Can we do any better?

*Exercise* 13. Explain why $P \wedge Q$ is equivalent to $\neg(\neg P \vee \neg Q)$ and $P \vee Q$ is equivalent to $\neg(\neg P \wedge \neg Q)$. Conclude that both $\wedge, \neg$ and $\vee, \neg$ are complete sets of connectives.

[Hint: You can check this by writing out the truth tables. But there's a shorter way using equivalences we've already seen.]

Can we do yet better and get it down to just one connective?

That you can do it with just $\neg$ is straightforward enough. $\neg$ only has one input, so there's way to combine information about different variables. Indeed, all you can express with $\neg$ is $P$ and $\neg P$ for an input variable $P$. (This is because e.g. $\neg\neg\neg P \equiv \neg P$ by the equivalence $\neg\neg P \equiv P$.) That we can't do it with just $\wedge$ is a little harder.

**Fact 15.** *The singleton set $\{\wedge\}$ is not complete. More specifically, if $\varphi$ is any formula using just $\wedge$ then the top row of the truth table, where all inputs have truth value 1, must have truth value 1.*

The key observation is that the only way to make a new formula using only $\wedge$ is to make a formula of the form $\varphi \wedge \psi$, where $\varphi$ and $\psi$ only use $\wedge$. So if you knew that $\varphi$ and $\psi$ satisfied this fact, the truth table for $\wedge$ immediately tells you that $\varphi \wedge \psi$ satisfies this fact.

Turning this into a formal proof uses the technique of *induction*. If you've seen induction in a previous class, a good exercise to explicitly write out this proof.[2]

*Exercise* 14. Make an analogous observation to explain why the singleton set $\{\vee\}$ is not complete. Do the same for $\{\Rightarrow\}$.

[2]If you haven't seen induction elsewhere: take Discrete Math in the spring. It's a good class!

The upshot of all this is that the smallest we can do with our basic connectives is a complete set of connectives with two elements.

If we want a single connective which is complete by itself we have to introduce a new connective. This can be done using NAND (= "not and") $\overline{\wedge}$.

| $P$ | $Q$ | $P \mathbin{\overline{\wedge}} Q$ |
|---|---|---|
| 1 | 1 | |
| 1 | 0 | |
| 0 | 1 | |
| 0 | 0 | |

From the truth table it is apparent that $P \wedge Q$ is equivalent to $\neg(P \mathbin{\overline{\wedge}} Q)$. Combining that observation with the fact that $\neg$ and $\wedge$ form a complete set of connectives, to prove that the singleton $\{\overline{\wedge}\}$ is complete we only need to prove that $\neg P$ can be expressed using only $\wedge$. Fortunately there are not many things to try.

| $P$ | $P \mathbin{\overline{\wedge}} P$ |
|---|---|
| 1 | |
| 0 | |

*Exercise* 15. Explicitly write out a formula using only $\overline{\wedge}$ which defines $P \wedge Q$. Can you explicitly write out a formula using only $\overline{\wedge}$ which defines $P \vee Q$?

Altogether, we have:

**Theorem 16.** *$\overline{\wedge}$ by itself forms a complete set of connectives.* □

*Exercise* 16. Write a truth table for NOR (= "not or") $\overline{\vee}$. Show that $\overline{\vee}$ by itself forms a complete set of connectives.

*Exercise* 17. Write definitions of all 16 possible binary connectives using $\wedge$, $\vee$, and $\neg$, where you make each as short as possible.

*Exercise* 18 (Extra Credit). Is $\{\oplus\}$ complete? Explain why or why not.

## 6. An application of boolean logic: circuits

An electronic circuit carries a signal—its voltage. By separating voltages into high (1) versus low (0) we can think of a circuit as carrying a truth value. It is possible to build switches that allow you to combine the signal (truth values) carried on the input circuits to determine the signal (truth value) of the output circuit. In this way, we can build *logic gates* which let us implement the connectives of boolean logic in electronic circuits.

By combining together many, many, many logic gates with many different circuits you can build an electronic digital computer. Everything done can be expressed by complicated rules about 0s and 1s, and if you go down to the hardware level it works by implementing the concepts of boolean logic into circuits.

This is something you could spend years studying. It's not the topic of this class. But I wanted to mention it as an application.

*Exercise* 19. A *multiplexer* is a device which selects one of multiple input values to be the output value. In its simplest form, a multiplexer chooses one of two binary inputs to be the outputs Namely, if the two inputs are $P$ and $Q$, each with value 0 or 1, then the value of the selector $S$ determines the output. If $S$ is 0 then the output is $P$, if $S$ is 1 then the output is $Q$.

Write a truth table which expresses the behavior of the multiplex $m(P, Q, S)$. Then write a formula using $\wedge$, $\vee$, and $\neg$ which gives this multiplex. Give a visual representation of your formula with connected input/output wires between the connectives.

## 7. From propositional to predicate logic

While our primary focus in this class will be propositional logics, it would remiss to say nothing about predicate logics. These extend propositional logic by introducing *predicates*. These are statements like "$p$ is a prime number" that include a variable. To say whether or not this statement is true, you need to specify what number the variable $p$ is standing for.

Predicate logic is useful because it lets you make statements about a whole domain of objects, such as "every differentiable function is continuous" or "some people are neither men nor women". These can be done by expanding our logical language to have *quantifiers*, which express something about how many things satisfy a predicate. The two quantifiers we will add are the universal quantifier $\forall$ and the existential quantifier $\exists$.

**Definition 17.** Let $P(x)$ be a predicate which can either be true or false of an object $x$. Then $\forall x\ P(x)$ means that $P(x)$ is true for every $x$ and $\exists x\ P(x)$ means that $P(x)$ is true for some $x$.

You have to be careful what you mean by "every" or "some". For example, you probably think it's true that every number is either even or odd—you could write this in predicate logic as $\forall x\ (E(x) \vee O(x))$. But if $x$ is allowed to be function on the real line or a rock then that wouldn't be true—a rock is neither even nor odd.

The resolution is that when we use quantifiers we must have a *domain of discourse* in mind. We're not quantifying over everything everywhere, we're just talking about a specific domain.

A full discussion of the semantics of quantifiers would require us to extend the notion of a valuation to predicates. Thereby we could formally define an evaluation function on formulas with quantifiers. For time reasons we will not do this, and our work with quantifiers will only require the intuitive definition above. Note, however, that when dealing with predicates we usually have some meaning in mind. So we don't want to just look at any way of saying whether $P(x)$ is true, but just what makes sense for the meaning.

Some predicates take more than one variable. Consider, for example, the predicate $M(x, y)$ expressing "$x$ is $y$'s mother". It wouldn't make sense to talk about this with just one variable.

*Exercise* 20. For each of the following sentences rewrite it in the form of a logical formula, using predicates for the atomic utterances. State the meaning of each predicate.

- Every differentiable function is continuous.
- There is a function which is continuous everywhere but differentiable nowhere.
- Everyone has a mother.
- Some people are neither men nor women.

*Exercise* 21. Convince yourself that the order of quantifiers matters by considering the two formulas $\forall x\ \exists y\ L(x, y)$ and $\exists y\ \forall x\ L(x, y)$ where $L(x, y)$ means "$x$ is larger than $y$". Write the meaning of these two formulas in ordinary language and explain why they have different meanings.

## 8. An omission: deductive calculuses

There is a big subject we won't touch on. Namely, with logical systems we don't just want to understand their semantics—roughly speaking, when things are true or false. We also want to know how they model reasoning. When can you conclude a formula $\varphi$ is true if you know $\psi$ is true? What are the rules for valid arguments?

This is an important topic. But this is a seven week class, so we have to omit something.

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