MATH 355 LECTURE NOTES
CHAPTER 4: TREES

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1. A first look at trees

In this chapter we are introduced to a species of object of interest in contemporary set theory: the tree.

Ordinals provide a yardstick by which to measure the lengths of transfinite iterative constructions. (More precisely, we've seen that the well-foundedness property of ordinals legitimizes the use of transfinite recursion.) But they are only for linear constructions—where each step has precisely one next step. We also want to talk about non-linear structures, able to have many branching paths.

At a useful level of generality is the notion of a partially ordered set. We saw this back in Chapter 0, and I repeat the definition now.

Definition 1. A *partially ordered set* (*poset*) is a set equipped with a partial order. Namely, a poset is $(P, \leq)$ so that $\leq$ is a binary relation on P satisfying.

- (Reflexivity) $p \leq p$ for all $p \in P$;
- (Antisymmetry) If $p \leq q$ and $q \leq p$ then $p = q$; and
- (Transitivity) If $p \leq q \leq r$ then $p \leq r$.

Example 2. Let X be any set. Then $(\mathcal{P}(X), \subseteq)$ is a partially ordered set.

Note that this example (unless $|X| \leq 1$) gives a poset with diamonds: If $a \neq b \in X$ then we have $\emptyset \subseteq \{a\}, \{b\} \subseteq \{a, b\}$ but $\{a\}$ and $\{b\}$ are incomparable—neither is a subset of the other.

Definition 3. Let $(P, \leq)$ be a poset and consider $p, q \in P$.

- p and q are *comparable* if either $p \leq q$ or $q \leq p$;
- p and q are *incomparable* if they are not comparable, i.e. if $p \not\leq q$ and $q \not\leq p$.

Observation 4. *A partial order is a linear order if and only if any two elements are comparable.* □

A particularly useful class of partially ordered sets don't have diamonds. These are the topic of this chapter.

Definition 5. An *tree* is a poset $(T, \sqsubseteq)$ so that for each $t \in T$ the set $\{s \in T : s \sqsubseteq t\}$ is well-ordered by $\sqsubseteq$. It's common to refer to elements of a tree as *nodes*.

Observation 6. *There are no diamonds in a tree $(T, \sqsubseteq)$: if $a \sqsubseteq b_0, b_1 \sqsubseteq c$ then b_0 and b_1 are comparable.*

Proof. By the tree property the set $\{t \in T : t \sqsubseteq c\}$ is well-ordered. So b_0 and b_1 are comparable. □

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Note that we only needed $\{s \in T : s \sqsubseteq t\}$ to be linearly ordered for this to work out. The reason to further require it to be well-founded is that the trees which see use in set theory have this property.

Example 7. Let $2^{<\omega}$ be the set consisting of finite binary strings. More formally, elements of $2^{<\omega}$ are functions $n \to 2$ for $n < \omega$. Order $2^{<\omega}$ by extension: $s \sqsubseteq t$ if $\operatorname{dom} s \subseteq \operatorname{dom} t$ and $t \upharpoonright \operatorname{dom} s = s$. Then $2^{<\omega}$ is a tree.

Example 8. Any well-order is a tree.

If you've done some graph theory, you might have seen the notion of a tree in that context (a tree being a connected graph without cycles). While they are related, the two notions aren't quite the same. Our notion is an order-theoretic one, where there is an ordering on the nodes. Whereas in graph theory there isn't a built-in order. That said, there are connections between the two notions, and some theorems can be ported from one context to the other.

Example 9. If S and T are trees then you can form a tree from their disjoint union by saying that any $s \in S$ and $t \in T$ are incomparable.

It's convenient to exclude examples like this.

Definition 10. A *root* of a tree $(T, \sqsubseteq)$ is a node $r \in T$ so that there is no $t \sqsubset r$ in T. If T has at most one root call it *rooted*.

Observation 11. *Any tree can be partitioned into disjoint rooted trees.*

Proof sketch. Let $(T, \sqsubseteq)$ be a tree. Define an equivalence relation $\sim$ on T as: $s \sim t$ iff there is r so that $r \sqsubseteq s, t$. Observe that $\sim$-equivalence classes are rooted trees. □

Consequently, if you want to understand trees it's usually sufficient to focus on rooted trees. You can then push theorems forward to multi-rooted trees appropriately.

Trees can be split into levels.

Definition 12. Let $(T, \sqsubseteq)$ be a tree and α be an ordinal. Then the *α-th level* of T, written T_α, is the collection of all $s \in T$ so that $\{t \in T : t \sqsubset s\}$ has ordertype α.

Definition 13. The *height* of a tree is the least α so that T_α is empty.

Proposition 14. *For any tree $(T, \sqsubseteq)$, we have*

$$T = \bigcup_{\alpha<\gamma} T_\alpha,$$

where γ is the height of T. Moreover, the levels are disjoint.

Proof. The moreover is immediate. For the rest, all we need to see is that if $\alpha > \gamma$ then T_α is empty. To that end, suppose otherwise that $s \in T_\alpha$. Enumerate $\{t \in T : t \sqsubset s\}$, in $\sqsubseteq$-increasing order, as $\langle t_i : i < \alpha \rangle$. Note that $\{t_i : i < \beta\}$ is the set of predecessors of t_β. So we get that t_γ is on level γ, which supposed to be empty. Contradiction. □

With well-orders, each element has exactly one successors. With trees elements can have many successors.

Definition 15. Let T be a tree and $t \in T$. A *successor* of t is a node $s \in T$ so that $t \sqsubseteq s$ and there is no node in-between. A node is a *leaf node* if it has no successors.

Observation 16. *If the height of T is a successor ordinal $\alpha + 1$, then every node $t \in T_\alpha$ is a leaf node.* □

Exercises.

(1) Draw a finite tree and label its levels.

2. Trees of height $\leq \omega$

We will start with these trees as they are both the most straightforward to understand and the ones that see the most use outside set theory.

Observation 17. *Let $(T, \sqsubseteq)$ be a tree. Then the height of T is $\leq \omega$ if and only if there is no limit γ with T_γ nonempty.* □

These trees are simpler to understand because they don't have limit levels to handle. Compare to how you can do induction on $\mathbb{N} = \omega$ with only the zero and successor cases, while general induction on an ordinal also needs the limit case.

A (rooted) tree of height $\leq \omega$ starts at a root, and then grows upward by nodes having successors. Maybe a node will be a leaf and the tree will stop growing along that path. Or it could be that the tree has no leafs, and the paths grow upward forever.

Definition 18. Let X be a set. Then $X^{<\omega}$ is the collection of all finite sequences from X. More precisely,

$$X^{<\omega} = \{f : n \to X \mid n < \omega\}.$$

Always, $X^{<\omega}$ is a tree of height ω under the *subsequence* relation: $s \sqsubseteq t$ if $\operatorname{dom} s \subseteq \operatorname{dom} t$ and $t \upharpoonright \operatorname{dom} s = s$.

Definition 19. Let X be a set. A *tree on* X is a tree $(T, \sqsubseteq)$ so that T is a subset of $X^{<\omega}$ which is closed under subsequence and $\sqsubseteq_T$ is the subsequence relation. That is, if $s \sqsubseteq t \in T$ then $s \in T$.

Later we will also look at trees of height $> \omega$, and so we will expand this definition. But for now let's only look at things of height $\leq \omega$. (We could have called this something like "a tree on X of height $\leq \omega$", but that's awful.)

Observe that if T is a tree on X then T is rooted, and its root is the empty function $0 \to X$.

Indeed, it turns out that any rooted tree is isomorphic to a tree of sequences.

Lemma 20. *Suppose $(T, \sqsubseteq)$ is a rooted tree of height $\leq \omega$. Then T is isomorphic to a tree on some set X.*

To be clear, isomorphism for trees is the usual definition of isomorphism for (partial) orders: S and T are isomorphic if there is a bijection $f : S \to T$ so that $s_0 \sqsubseteq_S s_1$ iff $f(s_0) \sqsubseteq_T f(s_1)$.

Proof. Morally, the isomorphism sends each element of T to the sequence of elements below it. But we have to do a little tweak to properly handle the root.

Let r be the root of T and set $X = T \setminus \{r\}$. Let's see that T is isomorphic to a tree on X. Given a non-root node t in T we associate the finite sequence $\langle t_1, t_2, \ldots, t_n \rangle$ so that $t_1 \sqsubset t_2 \sqsubset \cdots \sqsubset t_n = t$ enumerates the elements of T below t in $\sqsubseteq$-increasing order. Then, the map $f : T \to X^{<\omega}$ defined as $f(r) = \emptyset$ and $f(t) = \langle t_1, t_2, \ldots, t_n \rangle$ for non-root t is an embedding of trees. So T is isomorphic to the range of f. □

The lesson of this lemma is that it's enough to look at trees of sequences, rather than just any abstract tree.

When looking at trees of height ω, one feature we want to know about is when there is a branch through the tree—a sequence that gets all the way to the top.

Definition 21. Let T be a tree of height ω. A *(cofinal) branch* through T is a sequence $b = \langle t_n : n \in \omega \rangle$ so that t_0 is a root and t_{n+1} is a successor of t_n for every n.

Observation 22. *A branch could be equivalently defined as:* $b = \langle t_n : n \in \omega \rangle$ *is a branch of* T *if* $t_n \sqsubseteq t_{n+1}$ *and* $t_n \in T_n$ *for every* n.

Observation 23. *If* $\langle t_n : n < \omega \rangle$ *is a branch through* T *then no* t_n *from the branch is a leaf.*

The meaning of a branch is especially clear when looking at trees of sequences:

Proposition 24. *Let* $T \subseteq X^{<\omega}$ *be a tree on* X. *Then any branch through* T *is given by an infinite sequence* $b \in X^\omega$ *so that* $b \upharpoonright n \in T$ *for every* $n < \omega$.

Proof. By the original definition, a branch would be a sequence $\langle s_n : n < \omega \rangle$ of sequences s_n each of length n. Then $b = \bigcup_n s_n \in X^\omega$ gives the infinite sequence.

In the other direction, if $b \in X^\omega$ and $b \upharpoonright n \in T$ for every n, then setting $s_n = b \upharpoonright n$ we get that $\langle s_n : n < \omega \rangle$ is a branch. □

We will slightly abuse notation and just refer to this $b \in X^\omega$ as a branch. After all, it's interdefinable from a branch as in the original definition.

Example 25. Any infinite sequence $b \in X^\omega$ is a branch through $X^{<\omega}$.

Example 26. Consider the tree T on ω defined to be the closure under subsequence of the set containing the nodes

$$\ell_n = \left\langle n, \underbrace{0, 0, \cdots, 0}_{n \text{ many}} \right\rangle.$$

Then T has height ω—because ℓ_n is on level n—but T does not have any branches.

These examples illustrate that for an infinitely-branching tree it may or may not have branches. We can say more if we look at finitely-branching trees.

Definition 27. A tree T is *finitely-branching* if each node in T has finitely many successors. Otherwise it is *infinitely-branching*.

To avoid a possible misunderstanding: infinitely-branching doesn't mean that *every* node has infinitely many successors, just that some nodes do.

Theorem 28 (Kőnig's lemma)**.** *Let* T *be a finitely branching, rooted tree with infinitely many nodes. Then* T *has a branch.*

Proof. Because T is finitely branching and rooted T is isomorphic to a subtree of $\omega^{<\omega}$. For convenience we will assume T is a tree on ω.

We build a branch $b = \langle k_n : n < \omega \rangle$ on T by recursion. We will build b by constructing its finite approximations $s_n = b \upharpoonright n$. For the zero case we have $s_0 = \emptyset$. For the successor case, we have already built $s_n = \{k_0, \dots, k_{n-1}\}$ so that there are infinitely many nodes above k_{n-1}. Look at the successors of s_n in T. For each successor u let $T \upharpoonright u = \{t \in T : u \sqsubseteq t\}$. By inductive hypothesis $T \upharpoonright s_n$ is infinite. Because $T \upharpoonright s_n = \{s_n\} \cup \bigcup_u T \upharpoonright u$, by the pigeonhole principle we have that at least one $T \upharpoonright u$ is infinite. Pick, say, the leftmost u so that $T \upharpoonright u$ is infinite. (Meaning that the last number in u, the one being added onto the end of s_n, is as small as possible.) Then set $s_{n+1} = u$.

We can always continue the construction, so by the recursion theorem we get a branch. □

2.1. **An application of trees.** We've been doing a lot of abstract talk, so let's see an example to get a feel for why this is useful.

The following theorem is famous for its only known proof being a computer aided proof.

Definition 29. A *graph* is a collection of *vertices* connected by *edges*. A graph is *planar* if it can be drawn without edges crossing.

Definition 30. A *coloring* of a graph is an assignment of colors to vertices so that vertices joined by an edge don't have the same color.

Theorem 31 (Four color theorem)**.** *Any planar graph can be colored with at most four colors.*

We won't prove this, since the proof amounts to doing a proof by cases with thousands of cases. But we can prove that the four color theorem for finite graphs implies the four color theorem for countably infinite graphs.

Theorem 32. *If the four color theorem holds for finite planar graphs, then it holds for countable planar graphs.*

Proof. Consider a countably infinite planar graph G. We will define a tree T of partial colorings of G. Namely enumerate the nodes of G as $\langle n_k : k < \omega \rangle$, a node in T is a coloring in at most 4 colors of the nodes $n_0, \ldots, n_k$ so that adjacent nodes in get different colors. The order for the tree is defined as $s \sqsubseteq t$ if s colors a subgraph of what t colors and t assigns the same colors as s on that subgraph.

Note that this tree is finitely branching. The reason is, if $s \in T_k$ then the successors of s are the colorings which extend s by coloring the k-th node n_k. But there's only four colors to assign to the new node, so s has at most four successors. By the four color theorem for finite planar graphs, we know that each T_k is nonempty—there's someway to color the graph of k many nodes. In particular, T is infinite. So by Kőnig's theorem there's a branch b through the tree.

Finally, observe that the branch b gives a coloring of all of G. Namely, the color b gives to a node n_k is the color any sufficiently large partial piece of b assigns to n_k. □

This is a typical use of Kőnig's lemma. It allows us to stitch together partial solutions on finite domains to a total solution on the entire infinite domain.

A similar application is in Ramsey theory. If you have the finite version of Ramsey's theorem you can prove the infinite version using Kőnig's lemma. This is one of the problems for this chapter.

Exercises.

(1) How many branches does $2^{<\omega}$ have? What about $\omega^{<\omega}$? $1^{<\omega}$?
(2) Construct a tree of height ω with precisely $\aleph_0$ many branches.
(3) Explain why the recursive construction to prove Kőnig's lemma doesn't work for the tree in Example 26.

3. Trees of height $> \omega$

Having looked at trees of height $\leq \omega$, let's now look upward to the transfinite.

To start, let's get examples of trees of any ordinal height.

Definition 33. Let X be a set and α be an ordinal. Then, $X^{<\alpha}$ is the set of all sequences of elements of X of length $< \alpha$. More formally,

$$X^{<\alpha} = \{f : \beta \to X \mid \beta < \alpha\}.$$

Example 34. $X^{<\alpha}$, ordered by the subsequence relation, is a tree of height α. The β-th level of the tree consists of the sequences of length β.

Definition 35. Let T be a tree of height α. A *branch* through T is a sequence $b = \langle t_\beta : \beta \in \alpha \rangle$ so that $t_\beta \sqsubseteq t_{\beta+1}$ and $t_\beta \in T_\beta$ for every $\beta < \alpha$.

If $T = X^{<\alpha}$ and α is limit then any function $b : \alpha \to X$ is a branch through T.

Definition 36. Let X be a set. A *tree on* X is a tree $(T, \sqsubseteq)$ so that T is a subset of $X^{<\alpha}$, for some α, which is closed under subsequence and $\sqsubseteq_T$ is the subsequence relation. That is, if $s \sqsubseteq t \in T$ then $s \in T$.

With trees of height $\leq \omega$, being isomorphic to a tree on some set required only that the tree have a single root. When there are limit level nodes things are more complicated. The following example illustrates the issue.

Example 37. We define a tree T of height $\omega + 1$. The levels $< \omega$ are just $2^{<\omega}$. For level ω, for each $b : \omega \to 2$ there are two distinct nodes ℓ_b and r_b so that for each n we have $(b \upharpoonright n) \sqsubseteq \ell_b, r_b$.

This tree is not isomorphic to a tree of sequences on 2. The reason is, nodes at level ω would have to uniquely correspond to sequences $b \in 2^\omega$ but for each such b there are two distinct nodes at level ω it would have to correspond to.

With this counterexample in mind, let's give a definition that avoids it. Following typical notational practice in mathematics, we will use the word "normal" to refer to the objects that avoid the yucky counterexample.

Definition 38. A tree T is *normal* if for every limit γ less than the height of T we have have that for $s, t \in T_\gamma$ that $s = t$ if and only if for each $\alpha < \gamma$ and each $u \in T_\alpha$ that $u \sqsubseteq s$ iff $u \sqsubseteq t$.

If we think of 0 as a limit level this this definition guarantees that T must be rooted. After all the condition right of the "if and only if" is vacuous when $\gamma = 0$ and so there must be a unique root.

Lemma 39. *Suppose T is normal. Then T is isomorphic to a tree on some X.*

Proof. The idea is the same as with trees of height $\leq \omega$: the isomorphism sends each element of T to the sequence of elements below it, with a little tweak to properly handle the root. And the normality condition ensures this definition works at limit levels.

Let r be the root of T and set $X = T \setminus \{r\}$. We see that T is isomorphic to a tree on X. Consider a node $t \in T$ of successor level $\alpha > 0$. Consider the sequence $\langle t_i : 0 < i \leq \alpha \rangle$ given by $t_i \sqsubseteq t$ is in level i. Observe that if $s, t \in T_\alpha$ then $s = t$ iff $\langle s_i \rangle = \langle t_i \rangle$.

Now consider a node $t \in T$ of limit level $\gamma > 0$. Consider the sequence $\langle t_i : 0 < i < \gamma \rangle$ given by $t_i \sqsubseteq t$ is in level i. Then for $s, t \in T_\gamma$ we have $s = t$ iff $\langle s_i \rangle = \langle t_i \rangle$, by the normality property. (Note that if we consider 0 a limit level then this means we associate the unique root to the empty sequence.)

This is what we use to define the isomorphism. Define $f : T \to X^{<\omega}$ as $f(r) = \emptyset$ and $f(t) = \langle t_i \rangle$ otherwise. This gives an embedding of trees, so T is isomorphic to the range of f. $\square$

Exercises.

(1) Say that $C \subseteq T$ is a *chain* if C is linearly ordered by $\sqsubseteq_T$. Prove that a branch through T is exactly a chain C so that $C \cap T_\alpha$ is nonempty for all α less than the height of T.

4. The tree property

The question we now turn to is, to what extent does Kőnig's lemma generalize to the uncountable? We will see that this question takes us to the cutting edge of set theory.

Roughly speaking, Kőnig's lemma says that if a tree of height ω is small then it has a branch. Here's smallness refers to the degree of branching: each node branches $< \omega$ many times. This notion of smallness is not suitable to trees with nodes of limit height.

Example 40. Consider $T = 2^{<\omega_1}$. This tree is binary—each node has 2 successors. Nonetheless at level ω it has $2^{\aleph_0}$ many nodes.

While this tree has a lot of nodes at the same level, it nonetheless has a branch. If we want a wide tree with no branches we need to refine the construction.

Example 41. We define a tree T on 2 of height $\aleph_1$. At levels $\leq \omega$, it looks like $2^{\leq \omega}$. That is, $T_\alpha = 2^\alpha$ when $\alpha \leq \omega$. To define the tree above level ω, fix an injection $f : \omega_1 \setminus \omega \to 2^\omega$.[1] Now define the tree at levels $> \omega$ by putting, for each $b \in 2^\omega$,

$$b^\frown \left\langle \underbrace{0, 0, \ldots, 0}_{\leq \alpha \text{ many}} \right\rangle \in T_{\omega+\alpha}$$

whenever $f(\alpha) = b$. Here, $s^\frown t$ is the *concatenation* operation: it's the sequence you get by putting t on the end of s.

This tree has height $\aleph_1$, because if $\omega < \alpha < \omega_1$ then there's a node at level $\omega + \alpha$ which extends $f(\alpha)$. But it has no branch of length ω_1, because we set it up so that there's no longer node extending $f(\alpha)$.

Another way you could've described this tree is, T is the downward closure of the sequences $b^\frown z_{f^{-1}(b)}$, where z_α is the sequence consisting of α many 0s. That is, $s \in T$ iff there is some b so that $s \sqsubseteq b^\frown z_{f^{-1}(b)}$.

What these examples demonstrate is that if we want to generalize Kőnig's lemma then we need better notion of smallness than having a small amount of branching at each node.

Definition 42. Let κ be a cardinal. A *κ-tree* is a normal tree T so that the height of T is κ and $|T|_\alpha < \kappa$ for each $\alpha < \kappa$.

Definition 43. Let κ be a cardinal. A *κ-Aronszajn tree* is a κ-tree which has no branch. And an *Aronszajn tree* is an $\aleph_1$-Aronszajn tree.

Nachman Aronszajn (roughly pronounced "air-in-shine") was a Polish mathematician.

Definition 44. Let κ be a cardinal. The *tree property at κ*, abbreviated $\mathsf{TP}(\kappa)$, asserts that no κ-tree is κ-Aronszajn.

Unfolding the definitions a little bit: Kőnig's lemma says $\mathsf{TP}(\aleph_0)$ holds. So $\mathsf{TP}(\kappa)$ is the generalization of Kőnig's lemma to κ, and a κ-Aronszajn tree is a counterexample to the generalization of Kőnig's lemma.

The question is are, then there any κ-Aronszajn trees?

It turns out this is easiest to answer when κ is singular. Recall that κ is singular if there is a cardinal $\lambda < \kappa$ so that there is a sequence of ordinals $\langle \alpha_i : i < \lambda \rangle$ less than κ whose supremum is κ. The smallest such λ is called the *cofinality* of κ.

[1] If the continuum hypothesis holds, f can be a bijection, but we don't need this assumption.

Theorem 45. *Suppose κ is singular. Then there is a κ-Aronszajn tree and thus $\mathsf{TP}(\kappa)$ fails.*

Proof. Let's first see it for $\kappa = \aleph_\omega$, and then we'll see how to generalize the idea. We build an $\aleph_\omega$-tree T with no branch as follows. Start with the root and add countably many immediate successors, call them a_n for $n < \omega$. Now to each a_n extend the tree by attaching a chain of length $\aleph_n$. Then the height of T is $\sup_n \aleph_n = \aleph_\omega$. But there is no branch because any chain has length $\aleph_n$ for some n.

More generally, suppose κ is singular with cofinality $\lambda < \kappa$. Fix a cofinal sequence $\langle \alpha_i : i < \lambda \rangle$ in κ. Build T by adding λ many successors to the root, call them a_i for $i < \lambda$. Then extend a_i by attaching a chain of length α_i. Done $\square$

Consequently, this question is most interesting for *regular* κ, those infinite cardinals which are not singular. (Note: when mathematicians say "interesting" what we mean is "difficult".) The natural first κ to look at is $\kappa = \aleph_1$.

Theorem 46 (Aronszajn, 1934)**.** *There is an Aronszajn tree. Consequently, $\mathsf{TP}(\aleph_1)$ fails.*

Proof. We construct a tree on ω. Define $T \subseteq \omega^{<\omega_1}$ to consist of all sequences $\alpha \to \omega$ which are one-to-one. Observe that if b were a branch through T then $\cup b$ would be an injection $\omega_1 \to \omega$, which is impossible.

Note, however, that T is not an Aronszajn tree. For example, $|T_\omega| = 2^{\aleph_0}$ because there are $2^{\aleph_0}$ injections $\omega \to \omega$. (A Chapter 2 problem was to prove there are $2^{\aleph_0}$ bijections, whence there's the same cardinality of injections.) We will find a subtree S of T which is a κ-tree. Because T has no branches we get for free that S has no branches, and so S will be Aronszajn.

We will build S by recursion on ω_1. What we will do is, for each α, pick $s_\alpha \in T_\alpha$ so that (a) $\omega \setminus \operatorname{ran}(s_\alpha)$ is infinite and (b) if $\beta < \alpha$ then $s_\alpha \upharpoonright \beta =^* s_\beta$. Here, $=^*$ is "equality mod finite":

$$x =^* y \qquad \text{iff} \qquad \{i : x(i) \neq y(i)\} \text{ is finite.}$$

Assuming this recursion can be successfully carried out, set S to consist of all $t \in T_\alpha$, for $\alpha < \omega_1$, so that $t =^* s_\alpha$. Observe that S_α is countable. This is because there are $\aleph_0$ many finite subsets of α and for each finite $e \subseteq \alpha$ there only countably many ways to modify s_α on e to get a different function $\alpha \to \omega$. Thus, S will be Aronszajn.

It remains to see we can do the desired induction. We handle all finite cases, including the zero case, by setting s_n to be the identity map $n \to n$. The successor case is also easy. Given s_α pick $k \notin \operatorname{ran}(s_\alpha)$ and set $s_{\alpha+1} = s_\alpha \cup \{(\alpha, k)\}$. The substantive work is at the limit stage. Assume γ is limit and we have already built s_α for all $\alpha < \gamma$. Pick a cofinal sequence

$$\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots < \gamma$$

whose supremum is γ. For each n pick $t_n \in T_{\alpha_n}$ as follows:

- $t_0 = s_{\alpha_0}$. (Note: this is $=$, not $=^*$.)
- $t_{n+1} =^* s_{\alpha_{n+1}}$ so that $t_{n+1} \upharpoonright \alpha_n = t_n$

We can inductively do this using (b). Namely, $s_{\alpha_{n+1}} \upharpoonright \alpha_n =^* s_{\alpha_n} =^* t_n$ and so we can set $t_{n+1}(i) = s_{\alpha_{n+1}}(i)$ for each $i \geq \alpha_n$ and $t_{n+1}(i) = t_n(i)$ for each $i < \alpha_n$. Finally, set $t = \bigcup_{n \in \omega} t_n$.

Note that $t : \gamma \to \omega$ and t satisfies (b). But t might not satisfy (a), since in taking the union we may have grown the range to be cofinite. We will ensure (a) holds by modifying t.. Specifically, set s_γ to agree with t on $\gamma \setminus \{\alpha_n : n \in \omega\}$ and set $s_\gamma(\alpha_n) = t(\alpha_{2n})$. Then $t(\alpha_{2n+1})$ is disjoint from $\operatorname{ran}(s_\gamma)$ and so (a) is satisfied. And note that (b) continues to be satisfied because if $\alpha < \gamma$ we have only modified t finitely many times $< \alpha$. $\square$

Can we generalize this argument to larger cardinals? Suppose you want to run the same construction on ω_2. Now your nodes will be one-to-one functions $s : \alpha \to \omega_1$, for $\alpha < \omega_2$. This gives you a tree T with no branches. You still need to cut it down to S. For this, you need to change $=^*$ to mean $x =^* y$ iff $\{i : x(i) \neq y(i)\}$ is countable. It looks like this works, but there's a cardinal arithmetic catch. For the ω_1 argument, there were countably many $t \in T_\alpha$ so that $t =^* s_\alpha$. But for the ω_2 argument, you get $2^{\aleph_0}$ many $t \in T_\alpha$ so that $t^* = s_\alpha$. It's consistent with the axioms of set theory that $2^{\aleph_0} > \aleph_1$ and so this construction doesn't produce an $\aleph_2$-Aronszajn tree.

Nonetheless, if the continuum hypothesis holds then the cardinal arithmetic works out and this construction gives an $\aleph_2$-Aronszajn tree.

Theorem 47. *If* CH *then there is an* $\aleph_2$*-Aronszajn tree. More generally, if* $2^\kappa = \kappa^+$ *then there is a* κ^{++}*-Aronszajn tree.*

I shan't give the full proof of this, as the construction for $\aleph_1$ is fiddly enough. Indeed, this is sufficiently technical that I won't even assign it for the problem set :)

Accepting this sketch in place of a proof, the natural next question is, what if the continuum hypothesis fails. Can we say what happens then?

Theorem 48 (Mitchell, 1972)**.** *It is consistent that there is no* $\aleph_2$*-Aronszajn tree. That is, it is consistent to have* $\mathsf{TP}(\aleph_2)$.

Mitchell's construction is interesting because it uses a *large cardinal*. He starts with a *weakly compact* cardinal, a type of cardinal number which ZFC cannot prove the existence of. He then uses a version of Cohen's forcing (originally used to produce a model of ZFC where CH fails) to produce a model of $\mathsf{TP}(\aleph_2)$.

It turns out that a weakly compact cardinal exactly captures the strength of $\mathsf{TP}(\aleph_2)$. Jack Silver proved that if there is no $\aleph_2$-Aronszajn tree then you can thin out the universe of sets to get a model of set theory in which there is a weakly compact cardinal. (Indeed, the cardinal which was $\aleph_2$ in V will be the weakly compact cardinal in this thinner universe.) So what we get is a necessary use of large cardinals.

After Mitchell's and Silver's results, there has been regular progress of the question of which regular cardinals can have the tree property. Let me list a few.

Theorem 49. *The following are consistent with* ZFC*, starting from appropriate large cardinals.*

- *(Mitchell, 1972) The tree property at* $\aleph_2$.
- *(Abraham, 1983) The tree property at both* $\aleph_2$ *and* $\aleph_3$.
- *(Cummings and Foreman, 1998) The tree property at every* $\aleph_n$ *for* $2 \leq n < \omega$.
- *(Neeman, 2014) The tree property at every* $\aleph_n$ *for* $2 \leq n < \omega$ *and also at* $\aleph_{\omega+1}$.
- *(Sinapova and Unger, 2018) The tree property at both* $\aleph_{\omega^2+1}$ *and* $\aleph_{\omega^2+2}$ *while* $\aleph_{\omega^2}$ *is strong limit. (That is,* $2^\kappa < \aleph_{\omega^2}$ *for all* $\kappa < \aleph_{\omega^2}$*.)*

Note the years on these results. Improving upon Mitchell's result has been the work of decades, and takes us up to the current cutting edge in set theory. Indeed, the large cardinals used in the better and better results are more and more powerful. Let me also address the jump from around $\aleph_\omega$ up to $\aleph_{\omega^2}$ with the Sinapova–Unger result. For technical reasons, it is very hard to have the tree property at double successors of a strong limit. For example, the following question is still open.

Question 50. *Is it consistent to have the tree property at* $\aleph_{\omega+1}$ *and* $\aleph_{\omega+2}$ *when* $\aleph_\omega$ *is strong limit?*

Exercises.

(1) Contemplate having the tree property at every possible cardinal. Whoa.

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