MATH 355 PROBLEM SET
CHAPTER 0: COUNTABLE SETS

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Problem 1. [This problem uses concepts from real analysis, so if you have not taken that class you will probably find it challenging.] *Follow this outline to reproduce Cantor's original 1874 proof that* $\mathbb{R}$ *is not countable.*

1. *Suppose you have a sequence* $\langle [a_n, b_n] : n \in \mathbb{N} \rangle$ *of nested closed intervals getting smaller on both sides; that is,* $a_n < a_{n+1} < b_{n+1} < b_n$ *for all* n*. Prove that if* $X = \bigcap_{n \in \mathbb{N}} [a_n, b_n]$ *then any* $x \in X$ *must not be any* a_n *nor any* b_n*.*
2. *Also prove that* X *is nonempty.*
3. *Explain how to produce a sequence* $\langle [a_n, b_n] : n \in \mathbb{N} \rangle$ *of strictly nested closed intervals from a sequence* $\langle x_n : n \in \mathbb{N} \rangle$ *of real numbers so that any number in the intersection* X *must not be any* x_n*.*
4. *Conclude that* $\mathbb{R}$ *is not countable.*

Problem 2. [This problem builds on the previous one and also uses concepts from real analysis.] *Show that if* $\langle x_n : n \in \mathbb{N} \rangle$ *is* dense *in* $\mathbb{R}$*—meaning that for any open interval* (a, b) *there's some* $x_n \in (a, b)$*—then the intersection* X *you obtain has a single element. Explain why for any real number* y *there's a sequence* $\langle q_n : n \in \mathbb{N} \rangle$ *of rational numbers so that doing this process with that sequence produces* $X = \{y\}$*.*

Problem 3. *Prove the infinitary De Morgan's Laws:*

$$C \setminus \bigcup_{A \in \mathcal{X}} A = \bigcap_{A \in \mathcal{X}} C \setminus A$$

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Problem 4. *Prove that a countable union of countable sets is countable. That is, if* $A_0, A_1, \ldots, A_n, \ldots$ *are all countable sets, then so is*

$$\bigcup_{n \in \mathbb{N}} A_n.$$

Problem 5. *Consider the function* $p : \mathbb{N}^2 \to \mathbb{N}$ *defined as*

$$p(a, b) = \frac{(a + b)(a + b + 1)}{2} + b.$$

Prove that p *is a bijection onto* $\mathbb{N}$*. Prove that for any* n *there is a bijection* $p_n : \mathbb{N}^n \to \mathbb{N}$ *which is a polynomial.*

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Problem 6. *For an element x in a linear order $(X, \leq)$, the* successor *of x, if it exists, is the smallest $y \in X$ so that $y > x$. Prove that if $(X, \leq)$ is a linear order so that every element of X has a successor then $\mathbb{N}$ order embeds into X.*

Problem 7. *If $(X, \leq)$ is a linear order with a minimum and so that every element of X has a successor, then you can formulate a version of the $+1$ induction property for X. Let z denote the minimum of X and let $s(x)$ denote the successor of $x \in X$.*

- *($+1$ induction property) For any $Y \subseteq X$, if $z \in Y$ and for every $x \in X$ we have $x \in Y$ implies $s(x) \in Y$, then $Y = X$.*

Prove that if X has the $+1$ induction property then X is order isomorphic to $\mathbb{N}$.

Problem 8. *A linear order X is* dense *if given any $x < y$ from X there is $z \in X$ so that $x < z < y$. And X has no endpoints if it has neither a minimum nor a maximum. Prove that if X is a countable dense linear order without endpoints then X is order isomorphic to $\mathbb{Q}$.*

Problem 9. [This problem uses some terminology from graph theory.] *The Rado graph is the unique up to isomorphism countable graph (G, E) satisfying the following property:*

- *If A, B are finite and disjoint sets of vertices in G then there is a vertex $v \in G \setminus (A \cup B)$ so that $a \mathrel{E} v$ for all $a \in A$ but $b \not\mathrel{E} v$ for any $b \in B$.*

Prove that any countable graph embeds into the Rado graph, where a graph embedding is a function $f : (H, E) \to (G, E)$ so that if $v \mathrel{E_H} w$ if and only if $f(v) \mathrel{E_G} f(w)$.

Problem 10. [This problem uses some terminology from algebra.] *Prove that for any complex number z there is a countable algebraically closed field $K \subseteq \mathbb{C}$ which contains z.*

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