MATH 355 PROBLEM SET
CHAPTER 1: ORDINALS

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Problem 1. *Let A be a set. Consider the collection of all ordinal numbers of well-orders whose domain is a subset of A, ordered by $\leq$. Call this collection of ordinals $\aleph(A)$. Prove that there is no injection $\aleph(A) \to A$. As a corollary derive that there is a well-order whose domain is not countable.*

Problem 2. *Prove that a linear order satisfying induction implies it is well-founded. That is, consider a linear order $(X, \leq)$. Assume that for any $Y \subseteq X$ if (for all $x \in X$ we have [if every $y < x$ is in X then $x \in X$]), then $Y = X$. (Brackets added to make the logical structure clear.) Prove that X must be well-founded.*

Problem 3. *Prove that the definition of ordinal addition by transfinite recursion matches our first definition of ordinal addition.*

Problem 4. *Prove that the definition of ordinal multiplication by transfinite recursion matches our first definition of ordinal multiplication.*

Problem 5. *Use the recursive definition of ordinal arithmetic to prove these properties.*

$$\begin{aligned} 0 \cdot \alpha &= 0; \\ 1 \cdot \alpha &= \alpha; \\ (\alpha \cdot \beta) \cdot \gamma &= \alpha \cdot (\beta \cdot \gamma); \\ \alpha \cdot (\beta + \gamma) &= \alpha \cdot \beta + \alpha \cdot \gamma; \end{aligned}$$

Problem 6. *Prove these properties of ordinal exponentation.*

$$\begin{aligned} &0^\beta = 0, && \text{if } \beta \neq 0; \\ &1^\beta = 1; \\ &\alpha^\beta < \alpha^\gamma, && \text{if } 1 < \alpha \text{ and } \beta < \gamma; \\ &\alpha^\gamma \leq \beta^\gamma, && \text{if } \alpha \leq \beta. \end{aligned}$$

Problem 7. *Prove these properties of ordinal exponentiation:*

$$\begin{aligned} \alpha^{\beta+\gamma} &= \alpha^\beta \cdot \alpha^\gamma; \\ \alpha^{\beta \cdot \gamma} &= \left(\alpha^\beta\right)^\gamma. \end{aligned}$$

Problem 8. *Prove that every ordinal can be written uniquely in the form*

$$\omega^{\alpha_0} \cdot n_0 + \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k,$$

where k is finite, each n_0 is a finite ordinal, and $\alpha_0 > \alpha_1 > \cdots > \alpha_k$ are ordinals.

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Problem 9. *A function f on the ordinals is* increasing *if $\alpha \leq f(\alpha)$ and f is* continuous *if given any sequence $\langle \alpha_i : i \in \lambda \rangle$ we have*

$$f\left(\sup_{i\in\lambda} \alpha_i\right) = \sup_{i\in\lambda} f(\alpha_i).$$

Show that if $f : \mathrm{Ord} \to \mathrm{Ord}$ is increasing and continuous then there always is an ordinal α so that $f(\alpha) = \alpha$. Moreover show that if β is any ordinal we can always find such α to be larger than β.

Problem 10. [This problem uses some ideas from real analysis or topology. If you've taken neither of those classes you may find it challenging.]

A set $C \subseteq \mathbb{R}$ is called closed *if it contais all its accumulation points, where x is an* accumulation point *of C if there are points in $C \setminus \{x\}$ abitrarily close to x.*[1] *A set $P \subseteq \mathbb{R}$ is called* perfect *if it is equal to the set of its accumulation points. Follow this outline to prove that any closed set $C \subseteq \mathbb{R}$ can be decomposed into a disjoint union of a perfect set and a countable set.*

(1) *Define the* derivative *of a set $A \subseteq \mathbb{R}$ to be $A' = \{x \in \mathbb{R} : x \text{ is an accumulation point of } A\}$. Show that if A is closed then so is A'.*
(2) *Starting from a closed set C recursively define a sequence of closed sets C_α:*
   - *$C_0 = C$;*
   - *$C_{\alpha+1} = C'_\alpha$; and*
   - *If γ is limit, then $C_\gamma = \bigcup_{\alpha<\gamma} C_\alpha$.*

   Say why every C_α is closed and that if $C_\alpha = C_{\alpha+1}$ then C_α is perfect.
(3) *Argue that $C_\alpha \setminus C_{\alpha+1}$ is always countable. [Hint: First see that if $x \in X \setminus X'$ then there is an open interval $U \ni x$ so that $U \cap X = \{x\}$.]*
(4) *Finally, show that no matter what closed C you start with, there's a countable ordinal α so that $C_\alpha = C_{\alpha+1}$.*
(5) *Conclude that C is the union of a perfect set and a countable set.*

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[1] That is, for any distance $\varepsilon > 0$ there is $y \in C \setminus \{x\}$ so that $|x - y| < \varepsilon$. If you know another characterization of closed sets you're welcome to use it instead for this problem.