Math 211: Calculus II

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A brief history of the calculus

- (Before 17th century) Many different mathematicians around the globe (such as Archimedes of Syracuse and Madhava of Sangamagrama) use infinitesimals to make specific calculations—volumes, areas, lengths.
- (17th–18th centuries) Gottfried Wilhelm Leibniz and Isaac Newton independently invent a systematic framework for this species of calculation, and prove the fundamental theorem of calculus which explains the connection between derivatives and integrals. Later mathematicians build on their work and the modern discipline of calculus is born.

  But this framework faced criticism for its use of infinitesimals.
- (19th century) Karl Weierstrass and other mathematicians show how to redo calculus in a framework that doesn't use infinitesimals
- (1960s) Abraham Robinson gives a mathematically rigorous framework for calculus using infinitesimals.

The two big ideas from calculus I

The derivative

The integral

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$\frac{\mathrm{d}y}{\mathrm{d}x}$ is the slope of $y = f(x)$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{st}\left(\frac{\Delta y}{\Delta x}\right) \qquad (\Delta x \approx 0)$$
$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

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The Fundamental Theorem of Calculus says these two concepts are opposites of each other.

Infinitesimal and infinite numbers

The conceptual foundation we used for calculus is based on infinitely small numbers—infinitesimals.

- Extend the reals $\mathbb{R}$ to the hyperreals $\mathbb{R}^*$ by adding in infinitely small and infinitely large numbers.
- $\mathbb{R}$ and $\mathbb{R}^*$ have the same elementary properties.
- Any real function $f(x)$ extends to a function on hyperreals.
- $x \approx a$ means x and a are infinitely close—their distance $x - a$ is infinitesimal.
- Every finite hyperreal x is infinitely close to exactly one real a, its standard part $a = \operatorname{st}(x)$.
- Standard parts let us do limits, and thereby the rest of calculus:

$$\lim_{x \to a} f(x) = L \qquad \text{means} \qquad f(x) \approx L \text{ when } x \approx a \text{ but } x \neq a$$

What's coming up in calculus II?

- **More about integrals**
  - Applications of integrals
  - Advanced integration techniques
- **Infinite series**
  - Can make sense of the idea of adding together infinitely many numbers.
  - Functions such as e^x and $\sin x$ can be fruitfully understood as being sums of infinitely many terms.
- **Calculus in other coordinates**
  - Polar coordinates
  - Parametric coordinates
  - An introduction to what you'll see in vector calculus.